UM104 – Calculus

Unit-I

Aim:

To construct strong knowledge in Calculus and its application in real time

Objectives:

1. To Understand the various concepts of Differential Calculus
2. Use the derivatives to find maxima and minima of a functions involving two and three variables
3. Apply differentiation to find envelope, curvature and pedal equation of a curve.
4. Study in detail the topic on reduction formulae and Bernoulli's formula.

Outcomes:

1. Apply the definition of a Partial derivative of a function to differentiate a given function.
2. Understand the Maxima and Minima of functions of 2 and 3 independent variables
3. Use Reduction formula and Bernoulli's formula techniques to Evaluate integral values

Prerequisites:

Students should have a strong foundation in algebra and pre-calculus

Websites For Learning All About Calculus:

1. Calculus-help.com
2. Calculus Made Easier
3. Calculus Reference
4. Calculus.org
5. CalculusQuest
6. Dan the Tutor
7. e-Calculus
8. Graphics for the Calculus Classroom
10. The History Of Calculus
11. S.O.S. Math - Calculus
12. Visual Calculus

Partial differentiation

Introduction

Earlier we had studied the differentiation of one function of one variable. We shall now consider the differentiation of function of two or more independent variable with respect to an independent variable.

Let *f(x,y)* be the function of two variable *x* and *y*. Suppose we let only *x* vary while keeping *y* fixed say *y=k*, where *k* is a constant. The partial derivative of f with respect to *x* and denoted by $\frac{\partial f}{dx}$.

$$\frac{\partial f}{\partial x}(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Similarly we can define the partial derivative of f(x,y) with respect to y

$$\frac{\partial f}{\partial y}(x,y)=\lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$$

Higher derivatives

If *f* is a function of two variables then its partial derivatives f_x, and f_y are function of two variables. So that we consider their partial derivatives $f_{xx,}\ f_{yx,}$ *and* f_{yy} which are called second order partial derivatives If *z=f(x,y)*, we use the following notations

$$f_{xx,}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 z}{\partial x^2}$$

$$f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 z}{\partial y\partial x}$$

$$f_{yx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 z}{\partial x\partial y}$$

$$f_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 z}{\partial y^2}$$

Thus notation f_{xy} or $\frac{\partial^2 f}{\partial y\partial x}$ means we first differentiate f with respect to *x* and then with respect to *y*. whereas, in computing f_{yx} the order is reversed.

Homogenous Function

A function of *f(x,y)* of two independent variables *x* and *y* is said to be Homogenous function in *x* and *y* of degree *n* if $f(tx,ty) = t^n f(x,y)$ for any positive quantity *t* where *t* is independent of *x* and *y*

Example

suppose

$$f(x,y) = \frac{x^3 + y^3}{x + y}$$
$$f(tx,ty) = \frac{t^3(x^3 + y^3)}{t(x + y)}$$
$$= t^2 f(x,y)$$

f(x,y) is a homogenous function of degree 2 in *x* and *y*

Euler's Theorem on homogenous functions

If *f* is a homogenous function of degree *n* in *x* and *y* then

$$x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = nf$$

Example

If $u=(x-y)(y-z)(z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution

Given

$$u = (x - y)(y - z)(z - x)$$
$$\frac{\partial u}{\partial x} = (y - z)(z - x) - (x - y)(y - z)$$
$$\frac{\partial u}{\partial y} = (z - x)(x - y) - (z - x)(y - z)$$
$$\frac{\partial u}{\partial z} = (x - y)(y - z) - (x - y)(z - x)$$

Adding $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Example

If $r^2 = x^2 + y^2$ *then show that* $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$

Solution

$$r^2 = x^2 + y^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\left(\frac{\partial r}{\partial x}\right) = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r.1 - x\dfrac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - \dfrac{x^2}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3}$$

similarly

$$\frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3}$$

adding

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - x^2 + r^2 - y^2}{r^3}$$

$$= \frac{2r^2 - (x^2 + y^2)}{r^3}$$

$$= \frac{1}{r} \qquad \text{..............................(1)}$$

$$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r}\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right]$$

$$= \frac{1}{r}\left[\frac{x^2 + y^2}{r^2}\right]$$

$$= \frac{1}{r}\left[\frac{r^2}{r^2}\right]$$

$$= \frac{1}{r} \qquad \text{......................(2)}$$

From (1) and (2) $$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$$

youtube link:

https://www.youtube.com/watch?v=6tQTRlbkbc8

https://www.youtube.com/watch?v=8ZAucbZscNA

https://www.youtube.com/watch?v=AXqhWeUEtQU

Assignment

1. If $z=e^x(x\cos y-y\sin y)$, *show that* $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$
2. If $u=x^2(y-z)+y^2(z-x)+z^2(x-y)$, *show that* $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$
3. If $u=(x-y)^4+(y-z)^4+(z-x)^4$, *show that* $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

Total differentials

Suppose *z=f(x,y)* is a differential function of *x* and *y* where *x =g(t)* and *y=h(t)* are both differential of *t*. then *z* is a differential function of *t* and total differential coefficient of *z* is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

In general if u is a function of $x_1, x_2, x_3, \ldots.. x_n$ and each x_i is a function of *n* variable $t_1, t_2, t_3, \ldots.. t_n$ and if $\frac{du}{dt_i}$ $(i = 1, 2, 3, \ldots n)$ exist then

$$\frac{du}{dt_i} = \frac{\partial u}{\partial x_1}.\frac{dx_1}{dt_1} + \frac{\partial u}{\partial x_2}.\frac{dx_2}{dt_2} + \ldots\ldots + \frac{\partial u}{\partial x_n}.\frac{dx_n}{dt_i} \quad i = 1, 2, 3, \ldots\ldots.$$

Example 1:

If $z = x^2 + y^2$, $x = t^3$, $y = 1 + t^2$ Find $\frac{dz}{dt}$

Solution

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt} \\ &= 2x.3t^2 + 2y.2t \\ &= 6t^5 + 4(1+t^2)t \\ &= 6t^5 + 4t^3 + 4t\end{aligned}$$

Example 2:

If $u = x\sqrt{1+y^2}$, $x = te^{2t}$, $y = e^{-t}$ Find $\dfrac{dz}{dt}$

Solution

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$= \sqrt{1+y^2}\,(e^{2t} + 2te^{2t}) + \frac{x.y}{\sqrt{1+y^2}}(-e^{-t})$$

$$= \frac{(1+y^2)(e^{2t} + 2te^{2t}) - xye^{-t}}{\sqrt{1+y^2}}$$

$$= \frac{(1+e^{-2t})(e^{2t} + 2te^{2t}) - t}{\sqrt{1+e^{-2t}}}$$

$$= \frac{e^{2t} + 1 + 2te^{2t} + 2t - t}{\sqrt{1+e^{-2t}}}$$

$$= \frac{e^{2t}(1+2t) + (1-t)}{\sqrt{1+e^{-2t}}}$$

Example 3:

If $u = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ *and* $z = e^t \cos t$ Find $\dfrac{dz}{dt}$

Solution

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$= 2x.e^t + 2y(e^t \sin t + e^t \cos t) + 2z(e^t \cos t - e^t \sin t)$$

$$= 2e^t\left[x + y(\sin t + \cos t) + z(\cos t - \sin t)\right]$$

$$= 2e^t\left[e^t + e^t \sin^2 t + e^t \sin t \cos t + e^t \cos^2 t - e^t \sin t \cos t\right]$$

$$= 2e^t\left[e^t + e^t(\sin^2 t + \cos^2 t)\right]$$

$$= 2e^t.2e^t$$

$$= 4e^{2t}$$

Example 4:

Find $\frac{dz}{dt}$ if $u = x^3y^4z^2$ $where$ $x = t^2,\ y = t^3,\ z = t^4$

Solution

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$= 3x^2y^4z^2.2t + 4x^3y^3z^2.3t^2 + 2x^3y^4z.4t^3$$

$$= 3t^4t^{12}t^8.2t + 4t^6t^9t^8.3t^2 + 2t^6t^{12}t^4.4t^3$$

$$= 6t^{25} + 12t^{25} + 8t^{25}$$

$$= 26t^{25}$$

youtube link

https://www.youtube.com/watch?v=iN3_5g0A_u8

https://www.youtube.com/watch?v=regif6yP6pQ

https://www.youtube.com/watch?v=41iIxeFfufA

Assignment

1. If $u = x^2y + 3xy^4$ $where$ $x = e^t,\ y = \sin t$ Find $\frac{dz}{dt}$
2. Find $\frac{dz}{dt}$ $z = 6x^3 - 3xy + 2y^2$ $where$ $x = e^t,\ y = \cos t$
3. Find $\frac{du}{dt}$ $u = xy^2z^3$ $where$ $x = \sin t,\ y = \cos t,\ z = 1 + e^{2t}$

JACOBIANS

If u,v are function of two variable x and y then the Jacobian is defined by the determinant

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

If u,v,w are function of three variable x,y and z then the Jacobian is defined by the determinant

$$J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

Example 1

If $u=\frac{y^2}{2x},\ v=\frac{x^2+y^2}{2x}\quad find\ \frac{\partial(u,v)}{\partial(x,y)}$

Solution

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{-y^2}{2x^2} & \frac{y}{x}\\ \frac{x^2-y^2}{2x^2} & \frac{y}{x}\end{vmatrix}$$

$$=\frac{-y^3}{2x^3}-\frac{y(x^2-y^2)}{2x^3}$$

$$=\frac{-y}{2x}$$

Example 2

If $x = r\cos\theta \quad y = r\sin\theta \quad find \ \dfrac{\partial(x,y)}{\partial(r,\theta)}$

Solution

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Example 3

If $u = \dfrac{1}{x}, \ v = \dfrac{x^2}{y} \ \ and \ \ w = x + y + zy^2 \quad find \ \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\dfrac{1}{x^2} & 0 & 0 \\ \dfrac{2x}{y} & \dfrac{-x^2}{y^2} & 0 \\ 1 & 1+2zy & y^2 \end{vmatrix}$$

$$= \left(\frac{x^2y^2}{x^2y^2}\right)$$

$$= 1$$

Example 4

If $\quad x + y + z = u, \quad y + z = uv, \quad z = uvw \quad prove \ that \ \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2$

Solution

$$\begin{aligned}
&x+y+z=u\\
&x=u-(y+z)\\
&\quad =u-uv\\
&\quad =u(1-v)
\end{aligned}$$

$$\begin{aligned}
&y+z=uv\\
&y=uv-z\\
&\quad =uv-uvw\\
&z=uvw
\end{aligned}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

$$=\begin{vmatrix}1-v & -u & 0\\ v-vw & u-uw & -uv\\ vw & uw & uv\end{vmatrix}$$

$$=\begin{vmatrix}1-v & -u & 0\\ v & u & 0\\ vw & uw & uv\end{vmatrix} \quad R_2+R_3$$

$$=uv\left[u(1-v)+uv\right]$$

$$=u^2v$$

youtube link

https://www.youtube.com/watch?v=CbJDIMqTZdU

https://www.youtube.com/watch?v=YlpTkYjUbb8

Assignment

1. If $x+y=u \quad y=uv \quad find \ \frac{\partial(x,y)}{\partial(u,v)}$
2. If $u+v=x \ and \ u-v=y \quad find \ \frac{\partial(u,v)}{\partial(x,y)}$
3. $u=x, \quad v=x+y, \quad w=x+y+z \quad find \ \frac{\partial(u,v,w)}{\partial(x,y,z)}$
4. $u=\frac{yz}{x}, \quad v=\frac{zx}{y}, \quad w=\frac{xy}{z} \quad then \ show \ that \ \frac{\partial(u,v,w)}{\partial(x,y,z)}=4$

Maxima and Minima of functions of 2 and 3 independent variables

The first and second derivatives of a function of one variable can be used to derivative its maxima and minima. Similarly the first order and second order partial derivatives can be used to determine maxima and minima of a function of two variables.

Let $f(x,y)$be any function of two variables x and y, supposed to be continuous for all values of these variable in the neighbourhood of their values a and b respectively.

We say that $f(x,y)$ has a maximum value at $f(a,b)$, if $f(a,b) > (a+h, b+k)$

For all sufficiently small independent values of h and k, positive or negative. On the other hand $f(a,b)$ is said to be a minimum value of $f(x,y)$, if $f(a,b) < (a+h, b+k)$ for all sufficiently small independent values of h and k, positive or negative.

Theorem

The necessary conditions for the existence of a maxima and minima of $f(x,y)$ *at* $x=a$ and $y=b$ are $f_x(a,b)=0$ and $f_y(a,b)=0$ where $f_x(a,b)$ and $f_y(a,b)$ respectively denotes the values of

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x=a$ and $y=b$

Note

The above conditions are necessary but not sufficient for the existence of a maxima and minima of $f(x,y)$ at (a,b). such points (a,b) for which $f_x(a,b)=0$ and $f_y(a,b)=0$ are known as critical points or stationary points.

Theorem

Let $f_x(a,b)=0$ and $f_y(a,b)=0$

Let $r=f_{xx}(a,b)=0$,$s= f_{yx}(a,b)=0$ *and* $t= f_{yy}(a,b)$

Then (i) *if* $(rt-s^2)>0$ *and* $r>0$, $f(x,y)$ *is minimum at* (a,b)

(ii) *if* $(rt-s^2)>0$ *and* $r,0$, $f(x,y)$ *is maxnimum at* (a,b)

(iii) *if* $(rt-s^2)<0$ $f(x,y)$ *is neither maximum nor minimum at* (a,b)

And (iv) *if* $(rt-s^2)=0$ *the case is doubtful.*

Example

Find the maximum and minimum values of $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

Solution

$$f(x,y) = 2(x^2 - y^2) - x^4 + y^4$$
$$\frac{\partial f}{\partial x} = 4x - 4x^3; \qquad r = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$$
$$\frac{\partial f}{\partial y} = -4y + 4y^3 \; ; \quad t = \frac{\partial f^2}{\partial y^2} = -4 + 12y^2$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

For maximum or minimum values

$$\frac{\partial f}{\partial x} = 0 \;\; and \;\; \frac{\partial f}{\partial y} = 0$$
$$4x(1 - x^2) = 0 \;\; and \;\; -4y(1 - y^2) = 0$$
$$x = 0, \;\; x \pm 1 \qquad and \quad y = 0, \;\; y = \pm 1$$

The points for maxima or minima are (0,0), (0,1), (0,-1), (1,0), (-1,0), (1,1), (-1,-1).

At *(0,0)* ,*(1,1)*, *(-1,-1)*, $rt\text{-}s^2<0$ and therefore these points are saddle points.

At (0,1) and (0,-1), $rt\text{-}s^2>0$ and $r>0$

The function is minimum at (0,1) and (0,-1)

The minimum value = -1

At (1,0) and (-1,0), $rt\text{-}s^2>0$ and $r<0$

The function is maximum at (1,0) and (-1,0)

The maximum value =1

Methods of Lagrange's Multipliers

Example 1

Find the maxima and minima if any of the functions
$f(x, y) = 12xy - 3y^2 - x^2 \ \ subject\ to\ x + y = 16$

Solution

$$f(x, y) = 12xy - 3y^2 - x^2$$
$$g(x, y) = x + y - 16$$
consider
$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$
$$= 12xy - 3y^2 - x^2 - \lambda(x + y - 16)$$
$$\frac{\partial F}{\partial x} = 12y - 2x - \lambda$$
$$\frac{\partial F}{\partial y} = 12x - 6y - \lambda$$
$$\frac{\partial F}{\partial \lambda} = -(x + y - 16)$$
solve
$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0 \ \ and \ \ \frac{\partial F}{\partial \lambda} = 0$$
$$-2x + 12y - \lambda = 0 \qquad \text{....................(1)}$$
$$12x - 6y - \lambda = 0 \qquad \text{.....................(2)}$$
$$-(x + y - 16) = 0 \qquad \text{........................(3)}$$
$$(2) - (1) \Rightarrow \ 14x - 18y = 0$$
$$7x - 9y = 0$$
$$7x + 7y = 112$$
$$16y = 112$$
$$y = 7$$
$$when \quad y = 7, \ \ x = 9$$
$$\therefore (9, 7) \ is\, an\, exteremum \ \ po\text{int}$$
$$f(9, 7) = 12(63) - 3(49) - 81$$
$$= 528$$

(0,16) is a point satisfying the constraint

x+y=16

f(0,16)= -3(256)= -768

f(9,7)>f(0,16)

the function f(x,y) is a maximum at (9,7) and the maximum value is 528

Example 2

A rectangular box without a lid is to be made from $12m^3$ of cardboard. Find the volume of such a box.

Solution

Let *x,y,z* be the dimensions of the box and *g(x,y,z) =12*

V=xyz

Consider the function

$$F(x,y,z) = V(x,y,z) - \lambda g(x,y,z)$$

$$= xyz - \lambda(xy + 2yz + 2zx - 12)$$

$$\frac{\partial F}{\partial x} = yz - \lambda(y + 2z)$$

$$\frac{\partial F}{\partial y} = zx - \lambda(x + 2z)$$

$$\frac{\partial F}{\partial z} = xy - \lambda(2y + 2x)$$

$$\frac{\partial F}{\partial \lambda} = -(xy + 2yz + 2zx - 12)$$

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial \lambda} = 0$$

$$yz = \lambda(y + 2z) \quad ....(1)$$

$$zx = \lambda(x + 2z) \quad ....(2)$$

$$xy = \lambda(2x + 2y) \quad ....(3)$$

$$xy + 2yz + 2zx = 12 \quad ....(4)$$

$Multiply\ (1)\ by\ \ x, \quad (2)\ by\ \ y, \quad (3)\ by\ \ z$

$$xyz = \lambda(xy + 2xz) \quad ....(5)$$

$$xyz = \lambda(xy + 2yz) \quad ....(6)$$

$$xyz = \lambda(2xz + 2yz) \quad ....(7)$$

$we\ can\ easily\ note\ that\ \lambda \neq 0\ \ for\ \lambda = 0$

$$x = y = z = 0$$

$from\ (5)\ and\ (6)$

$$xy + 2xz = 2yz + xy$$

$$x = y \quad (\sin ce\ \ z \neq 0)$$

$from\ (6)\ \ and\ (7)\ we\ get\ \ y = 2z$

$$x = y = 2z$$

$from\ (4) \quad 4z^2 + 4z^2 + 4z^2 = 12$

$$12z^2 = 12$$

$$z^2 = 1$$

Since *x,y,z* are positive and *z=1* we get *x=2, y=2*

The maximum volume is $2(2)(1)=4m^3$

Assignment

(i) find the maxima and minima of the function $f(x,y)=3x^2+4y^2-xy \quad if \quad 2x+y=21$

(ii) find the extreme values of the function $f(x,y)=x^2+y \; on the circle \; x^2+y^2=1$

(iii) Using Lagrange's multipliers method find the maximum and minimum values of

$f(x,y)=x^2-y^2 \quad subject \; to \; x^2+y^2=1$

youtube link

https://www.youtube.com/watch?v=GoyeNUaSW08

https://www.youtube.com/watch?v=N_CyeSqqYs4

https://www.youtube.com/watch?v=vD2pflgcA6w

Unit 2

Curvature The rate of bending of a curve in any interval is called the curvature of the curve in that interval

Curvature of a circle The curvature of a circle at any point on it equals the reciprocal of its radius.

Radius of curvature The radius of curvature of a curve at any point on it is defined as the reciprocal of the curvature

Cartesian form of radius of curvature $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$

Parametric equation of radius of curvature $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

Implicit form of radius of curvature $\rho = \frac{(r^2+r'^2)^{3/2}}{(r^2+2r'^2-rr'')}$

Centre of curvature

The circle which touches the curve at P and whose radius is equal to the radius of curvature and its centre is known as centre of curvature.

Equation of circle of curvature $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

Centre of curvature $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$ & $\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$

Envelope A curve which touches each member of a family of curves is called envelope of that family curves.

Envelope of a family of curves The locus of the ultimate points of intersection of consecutive members of a family of curve is called the envelope of the family of curves.

Property:1 The normal at any point of a curve is a tangent to its evolute touching at the corresponding centre of curvature.

Property:2 The difference between the radii of curvature at two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.

Property:3 The envelope of a family of curves touches at each of its point. The corresponding member of that family.

Evolute as the envelope of normals The normals to a curve form a family of straight lines.we know that the envelope of the family of these normals is the locus of the ultimate points of intersection of consecutive normals.but the centre of curvature of a curve is also the point of consecutive normals.hence the envelope of the normals and the locus of the centres of curvature are the same that is ,the evolute of a curve is the envelope of the normals of the curve

1. **Find the radius of curvature of y=e^x at x=0**

Ans: $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

y=e^x

$y_1=e^x$ at x= 0 $y_1=1$

$y_2=e^x$ at x= 0 $y_2=1$

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \rho = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}$

2. **Find the radius of curvature of at x = $\frac{\pi}{2}$ on the curve y = 4 sin x – sin 2x**

Ans: $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

y_1=4 cosx – 2 cos 2x at x= $\frac{\pi}{2}$ $y_1=2$

y_2=4 sin x + sin 2x at x = $\frac{\pi}{2}$ $y_2=-4$

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \rho = \frac{(1+4)^{3/2}}{-4} = \frac{5\sqrt{5}}{2}$

3. Given the coordinates of the centre of curvature of the curve is given as $\bar{x} = 2a + 3at^2$ $\bar{y} = -2at^3$ Determine the evolute of the curve

Ans: $\bar{x} = 2a + 3at^2$ $t^2 = (\bar{x} - 2a/_{3a})$------------ 1

$\bar{y} = -2at^3$ $t^3 = \bar{y}/-2a$ ------------ 2

$(\bar{x} - 2a/_{3a})^3 = (\bar{y}/-2a)^2$

$4(\bar{x} - 2a)^3 = 27a\bar{y}^2$

The locus of the centre of curvature (evolute) is $4(x-2a)^3 = 27ay^2$

4. Write the envelope of $Am^2+Bm+C=0$, where m is the parameter and A,B and C are functions of x and y.

Solution: Given $Am^2+Bm+C=0$……………………(1)

Differentiate (1) partially w.r.t. 'm'

$2Am+B=0$ $m=-B/2A$…………(2)

Substitute (2) in (1) we get

$A(-B/2A)^2+B(-B/2A)+C=0$

$AB^2/4A^2-B^2/2A+C=0$

$AB^2-2AB^2+4A^2C=0$

$-AB^2+4A^2C=0$

Therefore $B^2-4AC=0$ which is the required envelope.

5. Find the radius of curvature at any point of the curve $y=x^2$.

Solution: Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given $y=x^2$ $y_1 = \frac{dy}{dx} = 2x$

$Y_2 = \frac{d^2y}{dx^2} = 2$

$$\rho = \frac{\left(1+(2x)^2\right)^{3/2}}{2}$$

$$= \frac{\left(1+4x^2\right)^{3/2}}{2}$$

6. Find the envelope of the family of straight lines x sin $\propto$ + y cos $\propto$ = p, $\propto$ being the parameter.

Solution: Given x sin $\propto$ + y cos $\propto$ = p……………. (1)

Differentiate (1) partially w.r.t. '$\propto$'

X cos $\propto$- y sin $\propto$ = o………………….(2)

Eliminate $\propto$ between (1) and (2)

X cos $\propto$ = y sin $\propto$

$$\frac{\sin\propto}{\cos\propto} = \frac{x}{y}$$

Tan $\propto = \frac{x}{y}$

Sin $\propto= \frac{x}{\sqrt{x^2+y^2}}$ cos $\propto= \frac{y}{\sqrt{x^2+y^2}}$

Substitute in (1)

x. $\frac{x}{\sqrt{x^2+y^2}}$ + y. $\frac{y}{\sqrt{x^2+y^2}}$ = p

$\sqrt{x^2+y^2}$ **= p**

Squaring on both sides, $x^2 + y^2 = p^2$ which is the required envelope

7. What is the curvature of $x^2 + y^2 - 4x - 6y + 10 = 0$ at any point on it .

Solution: Given $x^2 + y^2 - 4x - 6y + 10 = 0$

The given equation is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$

Here 2g =-4 g=-2

2f =-6 f=-3

Centre C(2,3), radius r =$\sqrt{g^2+f^2-c}$

r=$\sqrt{4+9-10}$

=$\sqrt{3}$

Curvature of the circle = $\frac{1}{r}$

There fore Curvature of $x^2+y^2-4x-6y+10=0$ is $\frac{1}{\sqrt{3}}$

8. Find the envelope of the family of straight lines y= mx$\pm\sqrt{m^2-1}$, where m is the parameter

Solution: Given y= mx$\pm\sqrt{m^2-1}$

$(y-mx)^2=m^2-1$

$Y^2+m^2x^2-2mxy-m^2+1=0$

$m^2(x^2-1)-2mxy+y^2+1=0$ which is quadratic in 'm'

Here,$A=x^2-1$ $B=-2xy$ $C=y^2+1$

The condition is $B^2-4AC=0$

$4x^2y^2-4(x^2-1)(y^2+1)=0$

$4x^2y^2-4x^2y^2-4x^2+4y^2+4=0$

$X^2-y^2=4$ which is the required envelope

9. Find the curvature of the curve $2x^2+2y^2+5x-2y+1=0$

Solution: Given $2x^2+2y^2+5x-2y+1=0$

÷2 $x^2+y^2+5/2x-y+1/2=0$

Here 2g =5/2 g=5/4

2f=-1 f=-1/2 centre C (-5/4,1/2) radius r= $\sqrt{g^2+f^2-c}$

$$=\sqrt{\frac{25}{16}+\frac{1}{4}-\frac{1}{2}}$$

$$=\sqrt{\frac{21}{16}}=\frac{\sqrt{21}}{4}$$

Therefore Curvature of the circle $2x^2 + 2y^2 + 5x - 2y + 1 = 0$ is $\frac{1}{r}=\frac{4}{\sqrt{21}}$

10. Define the curvature of a plane curve and what the curvature of a straight line.

Solution: The rate at which the plane curve has turned at a point (rate of bending of a curve is called the curvature of a curve. The curvature of a straight line is zero.

Solution: Given $X^2 + y^2 - 6x + 4y + 6 = 0$

The given equation is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$

Here 2g =-6 g=-3

2f =4 f=2

Centre C(3,-2), radius r =$\sqrt{g^2+f^2-c}$

$r=\sqrt{4+9-6}$

$=\sqrt{7}$

Radius of Curvature of the circle = radius of the circle=$\sqrt{7}$

11. Find the envelope of the family of circles $(x-\propto)^2+y^2=4\propto$,where $\propto$ is the parameter.

Solution: Given $(x-\propto)^2+y^2=4\propto$

$X^2-2\propto x+\propto^2-4\propto+y^2=0$

$\propto^2-2\propto(x+2)+x^2+y^2=0$ which is quadratic in $\propto$

The condition is $B^2-4AC=0$

Here A=1 B=-2 (x+2) C= x^2+y^2

$4(x+2)^2-4(x^2+y^2)=0$

$x^2-4x+4- x^2-y^2=0$

$y^2+4x=4$ which is the required envelope.

12. Find the envelope of the family of straight lines $y=mx+\frac{a}{m}$ for different values of 'm'.

Solution: Given $y=mx+\frac{a}{m}$

$m^2x-my+a=0$ which is quadratic in 'm'

The condition is $B^2-4AC=0$

Here A=x B=-y C=a

$Y^2-4ax=0$

There fore y^2=4ax which is the required envelope.

13. Find the envelope of the line $\frac{x}{t}$+yt=2c,where ‘t’ is the parameter.

Solution: Given $\frac{x}{t}$+yt=2c

Yt^2-2ct+x=0 which is quadratic in ‘t’

The condition is B^2-4AC=0

Here A=y B=-2c C=x

C^2-xy=0

There fore xy=c^2 which is the required envelope.

14. Find the radius of curvature of the curve y=c cosh(x/c)at the point where it crosses the y-axis.

Solution: Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given y=c cosh(x/c) and the curve crosses the y-axis. (i.e.)x=0 implies y=c.

Therefore the point of intersection is (0,c)

$\frac{dy}{dx}$**=c cos h(x/c)(1/c)=cos h (x/c)**

$\frac{dy}{dx}$**(0,c)=cos 0=1**

$\frac{d^2y}{dx^2}$**=cos h(x/c)(1/c)**

$\frac{d^2y}{dx^2}$**(0,c)=1/c**

$\rho = \frac{(1+1)^{3/2}}{1}$=c2$\sqrt{2}$

15. **Find the radius of curvature of the curve xy=c^2at (c,c).**

Solution: Radius of curvature $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$

Given $xy=c^2$

$\frac{dy}{dx}$=x$\frac{dy}{dx}$+y=0

$\frac{dy}{dx}$=-$\frac{y}{x}$ implies $\frac{dy}{dx}$(c,c)=-1

$\frac{d^2y}{dx^2}$=-$\left[\frac{x\frac{dy}{dx}-y.1}{x^2}\right]$

$\frac{d^2y}{dx^2}$(c,c)= =-$\left[\frac{c(-1)-c}{c^2}\right] = \frac{2c}{c^2} = \frac{2}{c}$

$$\rho = \frac{\left(1+(-1)^2\right)^{3/2}}{2/c} = \frac{c2\sqrt{2}}{2}$$

$$\rho = c\sqrt{2}$$

Find the radius of curvature at the point $(\boldsymbol{acos^3\theta}, \boldsymbol{asin^3\theta})$ on the curve $\boldsymbol{x^{2/3} + y^{2/3} = a^{2/3}}$.

Solution: Given x= $acos^3\theta$……………………(1)

Y= $asin^3\theta$…………………..(2)

Differentiate (1) and (2) w.r.t θ

$\frac{dx}{d\theta} = 3acos^2\theta(-sin\theta) = -3a\ sin\theta cos^2\theta$

$$\frac{dy}{d\theta} = 3asin^2\theta(cos\theta) = 3acos\theta sin^2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\cos\theta\sin^2\theta}{-3a\sin\theta\cos^2\theta} = -\tan\theta$$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}(-\tan\theta).\frac{d\theta}{dx}$

$=-\sec^2\theta.\frac{1}{-3a\sin\theta\cos^2\theta}$

$\frac{d^2y}{dx^2}=\frac{1}{3a\sin\theta\cos^4\theta}$

Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

$=\frac{(1+\tan^2\theta)^{3/2}}{\frac{1}{3a\sin\theta\cos^4\theta}} = 3a\sin\theta\cos^4\theta\,(\sec^2\theta)^{3/2}$

$=3a\sin\theta\cos^4\theta\sec^3\theta = 3a\sin\theta\cos\theta$

$$\rho = 3a\sin\theta\cos\theta$$

Find the radius of curvature of the curve $y^2 = x^2\frac{(a+x)}{(a-x)}$ at the point (-a,0).

Solution: Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Given $y^2 = x^2\frac{(a+x)}{(a-x)} = \frac{ax^2+x^3}{a-x}$

Differentiate w.r.t. 'x'

$2y\frac{dy}{dx} = \frac{(a-x)(2ax+3x^2)-(ax^2+x^3)(-1)}{(a-x)^2}$

$$\frac{dy}{dx} = \frac{(a-x)(2ax+3x^2)-(ax^2+x^3)(-1)}{2y(a-x)^2}$$

$$\frac{dy}{dx}(-a,0) = \frac{2a(-2a^2+3a^2)+(a^3-a^3)}{0} = \infty$$

$$\therefore \rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\boldsymbol{\frac{d^2x}{dy^2}}}$$

$$\frac{dx}{dy} = \frac{y(a-x)^2}{(a^2x+ax^2-x^3)} = \frac{0}{(a^2x+ax^2-x^3)} = 0$$

$$\boldsymbol{\frac{d^2x}{dy^2}} = \frac{(a^2x+ax^2-x^3)\left[y.2(a-x)\left(-\frac{dx}{dy}\right)+(a-x)^2.1\right]-y(a-x)^2\left[a^2\frac{dx}{dy}+2ax\frac{dx}{dy}-3x^2\frac{dx}{dy}\right]}{(a^2x+ax^2-x^3)^{\boldsymbol{2}}}$$

$$\boldsymbol{\frac{d^2x}{dy^2}(-a,o) = \frac{(-a^3+a^3+a^3)(4a^2)}{(-a^3+a^3+a^3)^2} = \frac{4a^5}{a^6} = \frac{4}{a}}$$

$$\therefore \rho = \frac{\{1+0\}^{3/2}}{\boldsymbol{\frac{4}{a}}} = \frac{a}{4}$$

Find the radius of curvature at the point (a,0)on the curve $\boldsymbol{xy^2 = a^3 - x^3}$.

Solution: Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Given $xy^2 = a^3 - x^3$

Differentiate w.r.t.'x'

$2xy\frac{dy}{dx}+y^2.1=-3x^2$ $\quad \frac{dy}{dx} = \frac{-3x^2-y^2}{2xy}$………………(1)

$$\frac{dy}{dx}(a,0) = \frac{-3a^2-0}{2a.0} = \infty$$

Therefore $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$

$$\frac{dx}{dy} = \frac{2xy}{-3x^2-y^2} \dots\dots\dots.(1)$$

$$\frac{dx}{dy}(a,0) = \frac{2a.0}{-3a^2-0} = 0$$

Differentiate (2) w.r.t.'y'.

$$\frac{d^2x}{dy^2} = \frac{2\left[(-3x^2-y^2)\left(x.1+y.\frac{dx}{dy}\right)-xy\left(-6x\frac{dx}{dy}-2y\right)\right]}{(-3x^2-y^2)^2}$$

$$\frac{d^2x}{dy^2}(a,0) = \frac{2[(-3a^2-0)(a+0)-0]}{(-3a^2-0)^2} = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$$

Therefore radius of curvature $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}} = \frac{\{1+0\}^{3/2}}{-2/3a} = \frac{-3}{2}a$

$\rho = \frac{3}{2}a$ (since the radius of curvature is non-negative)

Find the curvature of the parabola y^2=4x at the vertex.

Solution: Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Given; y^2=4x

Differentiate w.r.t.'x'

$2y\,\frac{dy}{dx} = 4$

$\frac{dy}{dx}$ =2/y

$$\frac{dy}{dx}(0,0) = \frac{2}{0} = \infty$$

Therefore $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$

$$\frac{dx}{dy} = \frac{y}{2} \ldots\ldots\ldots (1)$$

$$\frac{dx}{dy}(0,0) = 0$$

Differentiate (1) w.r.t.'y'.

$$\frac{d^2x}{dy^2} = \frac{1}{2}$$

Therefore $\rho = \frac{\{1+0\}^{3/2}}{1/2}$=2

Curvature K=1/ ρ =1/2

Find the radius of curvature of the curve 27ay^2= 4x^3 at the point where the tangent of the curve makes an angle 45^0 with the X- axis.

Solution; Let (x_1,y_1) be the point on the curve at which the tangent makes an angle 45^0 with the X- axis.

$\frac{dy}{dx}$(x_1,y_1) =Tan 45^o=1--------------------------------- (1)

Given 27ay^2= 4x^3

Differentiate w.r.t.'x'

54ay$\frac{dy}{dx}$=12x^2 $\frac{dy}{dx}$=$\frac{2x^2}{9ay}$

$\frac{dy}{dx}$(x_1,y_1) =$\frac{dy}{dx}$=$\frac{2x_1^2}{9ay_1}$-----------------------------------(2)

$\frac{dy}{dx}$(x_1,y_1) =Tan 45^o=1= $\frac{2x_1^2}{9ay_1}$

Gives $y_1 = \frac{2x_1^2}{9a}$-----------------------------(3)

As (x_1,y_1) lies on the curve 27ay^2_1= 4x^3_1 ---------------------------------(4)

Using $y_1 = \frac{2x_1^2}{9a}$ gives x_1= 3a

And using (3) gives y_1= 2a

Y_1 at (3a,2a)= 1

Y_2= $\frac{2}{9a}\left[\frac{y.2x-x^2.y_1}{y^2}\right]$

$Y_2 = \frac{2}{9a}\left[\frac{2.3a.2a-9a^2.1}{4a^2}\right]$=1/6a

Therefore radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1/6a}$

$$\rho = 12a\sqrt{2}$$

Find the equation of the envelope of $\frac{x}{a}+\frac{y}{b}=1\ where\ a^2+b^2=c^2$.

Solution: Given that $\frac{x}{a}+\frac{y}{b}=1$…………………(1)

And $a^2+b^2=c^2$……………..(2)

Differentiate (1)and(2) w.r.t 'b'

$\frac{-x}{a^2}\frac{da}{db}-\frac{y}{b^2}=0$………………..(3)

$2a\frac{da}{db}+2b=0$…………………………(4)

(3)gives $\frac{da}{db}=-\frac{a^2y}{b^2x}$……………..(5)

(4)gives$\frac{da}{db}=\frac{-b}{a}$……………………(6)

From (5)and (6) $\frac{-b}{a}=-\frac{a^2y}{b^2x}$

$$\frac{x}{a^3}=\frac{y}{b^3}=\frac{x/a}{a^2}=\frac{y/b}{b^2}=\frac{\frac{x}{a}+\frac{y}{b}}{a^2+b^2}=\frac{1}{c^2}$$

$$\frac{x}{a^3}=\frac{1}{c^2}\,and\,\frac{y}{b^3}=\frac{1}{c^2}$$

$$a=(xc^2)^{1/3}\ and\ b=(yc^2)^{1/3}$$

Substitute in (2) we get, $(xc^2)^{2/3}+(yc^2)^{2/3}=c^2$

Therefore $x^{2/3}+y^{2/3}=c^{2/3}$ which is the required envelope.

Find the equation of circle of curvature of the parabola y2=12x at the point (3,6).

Solution: The equation of circle of curvature is $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

Where, $\bar{x}=x-\frac{y_1}{y_2}(1+y_1^2)$

$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

Given $y^2=12x$

Differentiate w.r.t.'x' we get

$2y\frac{dy}{dx}=12$ implies $\frac{dy}{dx} = \frac{6}{y}$

$Y_1=\frac{dy}{dx}(3,6)=1$ $\frac{d^2y}{dx^2} = \frac{-6}{y^2}\frac{dy}{dx}$

$Y_2=\frac{d^2y}{dx^2}(3,6)=-1/6$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{-1/6} = -12\sqrt{2}$$

$\rho = 12\sqrt{2}$ (ρ can not be negative)

$$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$$

$=3-\frac{1}{-1/6}(1+1) = 15$

$$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2) = 6 + \frac{1}{-1/6}(1+1) = -6$$

Therefore,the equation of circle of curvature is $(x-15)^2 + (y+6)^2 = 288$

Find the radius of curvature at 't' on x=e^{t}cost,y=e^{t}sint.

Solution: Radius of curvature $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

Given $x = e^t cost\, , y = e^t sint$

X'$=\frac{dx}{dt} = e^t cost - e^t sint = e^t(cost - sint)$

Y'$=\frac{dy}{dt} = e^t cost + e^t sint = e^t(cost + sint)$

X''$=\frac{d^2x}{dt^2} = e^t(-sint - cost) + e^t(cost - sint) = -2e^t sint$

Y''$=\frac{d^2y}{dt^2} = e^t(-sint + cost) + e^t(cost + sint) = 2e^t cost$

$\Rightarrow\therefore$ The radius of curvature is $\rho = \frac{(x\prime^2 + y\prime^2)^{3/2}}{x'y'' - y\prime x\prime\prime}$

$$\rho = \frac{([e^t(cost - sint)]^2 + [e^t(cost + sint)]^2)^{3/2}}{e^t(cost - sint).2e^t cost - e^t(cost + sint).(-2e^t sint)}$$

$=\frac{(e^{2t}[cos^2t + sin^2t - 2sintcost + cos^2t + sin^2t + 2sintcost])^{3/2}}{2e^{2t}[cos^2t - sintcost + sintcost + sin^2t]} = \frac{(2e^{2t})^{3/2}}{2e^{2t}} = \sqrt{2}e^t$

$\therefore\ \rho = \sqrt{2}e^t.$

Find the envelope of $\frac{x}{l}+\frac{y}{m}=1$ **where l and m are connected by** $\frac{l}{a}+\frac{m}{b}=1$ **and a,b are constants.**

Solution: Given that $\frac{x}{l}+\frac{y}{m}=1$………………..(1)

$\frac{l}{a}+\frac{m}{b}=1$…………………(2)

Differentiating (1) w.r.t.'m'

$$x\left(\frac{-1}{l^2}\right)\frac{dl}{dm}+y\left(\frac{-1}{m^2}\right)=0$$

$$\frac{dl}{dm}=\frac{-yl^2}{xm^2}\text{…….(3)}$$

Differentiating (2) w.r.t.'m'

$$\frac{l}{a}\frac{dl}{dm}+\frac{1}{b}=0$$

$$\frac{dl}{dm}=\frac{-a}{b}\text{…………..(4)}$$

From (3) and (4)

$$\frac{-yl^2}{xm^2}=\frac{-a}{b}\Rightarrow\frac{by}{m^2}=\frac{ax}{l^2}$$

$$\frac{\frac{y}{m}}{\frac{m}{b}}=\frac{\frac{x}{l}}{\frac{l}{a}}=\frac{\frac{y}{m}+\frac{x}{l}}{\frac{m}{b}+\frac{l}{a}}=1$$

$\frac{by}{m^2}=1,\frac{ax}{l^2}=1\Rightarrow m=\sqrt{by},l=\sqrt{ax}$ substitute in equation (2) ,

$$\frac{\sqrt{ax}}{a} + \frac{\sqrt{by}}{b} = 1$$

$\Rightarrow \sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ which is the required envelope.

Find the points on the parabola $y^2 = 4x$ at which the radius of curvature is $4\sqrt{2}$.

Solution: Given $y^2 = 4x$……………(1)

Let, P (a,b) be the point on the curve $y^2 = 4x$ at where $\rho = 4\sqrt{2}$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

Differentiate (1) w.r.t. 'x'

Y_1=2y$\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$$\frac{dy}{dx}(a, b) = \frac{2}{b}$$

Y_2=$\frac{d^2y}{dx^2} = -\frac{2}{y^2}\frac{dy}{dx}$

$$\frac{d^2y}{dx^2}(a, b) = -\frac{4}{b^3}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(4 + b^2)^{3/2}}{4} = 4\sqrt{2}$$

But, $b^2 = 4a \Rightarrow \frac{(4+4a)^{3/2}}{4} = 4\sqrt{2}$

$8(1 + a)^{3/2} = 16\sqrt{2} \Rightarrow (1 + a)^3 = 2^3$

a+1=2⇒a=1, $b^2 = 4 \Rightarrow b = \pm 2$

∴The points are (1,2),(1,-2)

Unit - III

Evolute

From the definition of centre of curvature we observe that for different points on the curve we get different centres of curvature. As the point on the curve vary the centre of curvature also vary. The locus of the centre of curvature of the given curve is called the evolute of the curve. The given curve is called the involute of the curve. It may be noted that while the given curve has a unique evolute, the evolute may have a family of involutes.

Problems

(1) Find the evolute of the parabola $y^2=4ax$.

Soln

The parametric eqn of parabola $y^2=4ax$ is

$$x = at^2 \qquad y = 2at$$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right)$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right)\cdot\frac{dt}{dx}$$

$$= -\frac{1}{t^2}\cdot\frac{1}{2at}$$

$$y_2 = -\frac{1}{2at^3}$$

Let $\bar{x}, \bar{y}$ be the centre of curvature then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}}$$

$$= at^2 + \frac{1}{t}\left(\frac{t^2+1}{t^2}\right) 2at^3$$

$$= at^2 + 2a + 2at^2$$

$$\boxed{\bar{x} = 2a + 3at^2}$$

$$t^2 = \frac{\bar{x} - 2a}{3a}$$

$$t = \left(\frac{x-2a}{3a}\right)^{1/2}$$

$$t^3 = \left(\frac{x-2a}{3a}\right)^{3/2}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}}$$

$$= 2at - \left(\frac{t^2+1}{t^2}\right) \cdot 2at^3$$

$$= 2at - 2at - 2at^3$$

$$\boxed{\bar{y} = -2at^3}$$

Now eliminating t b/w the above

$$\bar{y} = -2at^3$$

$$= -2a\left(\frac{\bar{x} - 2a}{3a}\right)^{3/2}$$

$$\bar{y}^2 = 4a^2\left(\frac{\bar{x}-2a}{3a}\right)^3$$

$$\bar{y}^2 = 4a^2 \frac{(\bar{x}-2a)^3}{27a^3}$$

$$27a\bar{y}^2 = 4(\bar{x}-2a)^3$$

$$\Rightarrow 27ay^2 = 4(x-2a)^3$$

---

(2) find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Soln

The parametric eqn is

$x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(-\frac{b}{a}\cot\theta\right)$$

$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{b}{a}\operatorname{cosec}^2\theta\cdot\frac{1}{-a\sin\theta}$$

$$y_2 = -\frac{b}{a^2}\operatorname{cosec}^3\theta$$

Let $(\bar{x},\bar{y})$ be the centre of curvature then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos\theta - \frac{\left(-\frac{b}{a}\cot\theta\right)\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= a\cos\theta - \frac{1}{a}\frac{\cos\theta}{\sin\theta}\left(a^2+b^2\frac{\cos^2\theta}{\sin^2\theta}\right)\sin^3\theta$$

$$a\bar{x} = a^2\cos\theta - \cos\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right)$$

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$$a\bar{x} = a^2\cos\theta - \cos\theta\left(a^2(1-\cos^2\theta)+b^2\cos^2\theta\right)$$

$$= a^2\cos\theta - a^2\cos\theta + a^2\cos^3\theta - b^2\cos^3\theta.$$

$$a\bar{x} = (a^2-b^2)\cos^3\theta.$$

$$\cos^3\theta = \frac{a\bar{x}}{a^2-b^2}$$

$$\boxed{\cos\theta = \left(\frac{a\bar{x}}{a^2-b^2}\right)^{1/3}}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= b\sin\theta + \frac{1+\frac{b^2}{a^2}\cot^2\theta}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}.$$

$$= b\sin\theta - \frac{1}{b}\left(a^2+\frac{b^2\cos^2\theta}{\sin^2\theta}\right)\sin^3\theta.$$

$$b\bar{y} = b^2\sin\theta - \sin\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right)$$

$$= b^2\sin\theta - \sin\theta\left(a^2\sin^2\theta + b^2(1-\sin^2\theta)\right)$$

$$= b^2\sin\theta - a^2\sin^3\theta - b^2\sin\theta + b^2\sin^3\theta$$

$$b\bar{y} = -(a^2-b^2)\sin^3\theta.$$

$$\sin^3\theta = \frac{b\bar{y}}{-(a^2-b^2)}$$

$$\boxed{\sin\theta = (-1)^{1/3}\left(\frac{b\bar{y}}{a^2-b^2}\right)^{1/3}}$$

$$\left(\frac{a\bar{x}}{a^2-b^2}\right)^{2/3} + \left(\frac{b\bar{y}}{a^2-b^2}\right)^{2/3} = \sin^2\theta + \cos^2\theta.$$

$$\left(\frac{a\bar{x}}{a^2-b^2}\right)^{2/3} + \left(\frac{b\bar{y}}{a^2-b^2}\right)^{2/3} = 1$$

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$$

$$\Rightarrow \quad (ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$$

---

③ Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Soln

The parametric equation of hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$x = a\sec\theta \qquad y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta \qquad \frac{dy}{d\theta} = b\sec^2\theta$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

$$= \frac{b/\cos\theta}{a\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{b}{a\sin\theta}$$

$$\boxed{y_1 = \frac{b}{a}\operatorname{cosec}\theta.}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{b}{a}\operatorname{cosec}\theta\right)$$

$$= \frac{d}{d\theta}\left(\frac{b}{a}\operatorname{cosec}\theta\right)\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\operatorname{cosec}\theta\cot\theta \cdot \frac{1}{a\sec\theta\tan\theta}$$

$$= -\frac{b}{a^2} \cdot \frac{1}{\sin\theta}\cot\theta \cdot \cos\theta\cot\theta$$

$$\boxed{y_2 = -\frac{b}{a^2}\cot^3\theta.}$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\sec\theta - \frac{\frac{b}{a}\operatorname{cosec}\theta\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{-\frac{b}{a^2}\cot^3\theta.}$$

$$= a\sec\theta + \frac{1}{a}\cdot\frac{1}{\sin\theta}\left(a^2+\frac{b^2}{\sin^2\theta}\right)\cdot\frac{\sin^3\theta}{\cos^3\theta}$$

$$a\bar{x} = a^2(\sec\theta) + (a^2\sin^2\theta + b^2)\cdot\frac{1}{\cos^3\theta}.$$

$$= a^2\sec\theta + (a^2(1-\cos^2\theta)+b^2)\cdot\frac{1}{\cos^3\theta}$$

$$= a^2\sec\theta + a^2\sec^3\theta - a^2\sec\theta + b^2\sec^3\theta.$$

$$a\bar{x} = (a^2+b^2)\sec^3\theta.$$

$$\sec^3\theta = \frac{a\bar{x}}{a^2+b^2}$$

$$\sec\theta = \left(\frac{a\bar{x}}{a^2+b^2}\right)^{1/3}$$

$$\boxed{\sec^2\theta = \left(\frac{a\bar{x}}{a^2+b^2}\right)^{2/3}}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= b\tan\theta + \frac{1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta}{\frac{-b}{a^2}\cot^3\theta}$$

$$= b\tan\theta + \frac{1}{b}\left(a^2+\frac{b^2}{\sin^2\theta}\right)\cdot\frac{\sin^3\theta}{\cos^3\theta}$$

$$= b\tan\theta - \frac{1}{b}\left(a^2+\frac{b^2}{\sin^2\theta}\right)\frac{\sin^3\theta}{\cos^3\theta}.$$

$$b\bar{y} = b^2\tan\theta - \left(a^2\frac{\sin^3\theta}{\cos^3\theta} + b^2\frac{\sin\theta}{\cos^3\theta}\right)$$

$$= b^2\tan\theta - a^2\tan^3\theta - b^2\tan\theta\sec^2\theta$$

$$= b^2\tan\theta - a^2\tan^3\theta - b^2\tan\theta(1+\tan^2\theta)$$

$$= b^2\tan\theta - a^2\tan^3\theta - b^2\tan\theta - b^2\tan^3\theta.$$

$$b\bar{y} = -(a^2+b^2)\tan^3\theta.$$

$$\tan^3\theta = \frac{b\bar{y}}{-(a^2+b^2)}$$

$$\tan\theta = (-1)^{1/3}\left(\frac{b\bar{y}}{a^2+b^2}\right)^{1/3}$$

$$\tan^2\theta = \left(\frac{b\bar{y}}{a^2+b^2}\right)^{2/3}$$

We know that

$$\sec^2\theta - \tan^2\theta = 1.$$

$$\left(\frac{a\bar{x}}{a^2+b^2}\right)^{2/3} - \left(\frac{b\bar{y}}{a^2+b^2}\right)^{2/3} = 1$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}$$

$$\Rightarrow (ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}.$$

(4) Find the evolute of $x^{2/3}+y^{2/3}=a^{2/3}$

Soln

The parametric eqn of $x^{2/3}+y^{2/3}=a^{2/3}$

$$x=a\cos^3\theta \qquad y=a\sin^3\theta$$

$$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$$

$$y_1=\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$\boxed{y_1=-\tan\theta}$$

$$y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{dx}(-\tan\theta)$$

$$=\frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}.$$

$$=-\sec^2\theta\cdot\frac{1}{-3a\cos^2\theta\sin\theta}.$$

$$\boxed{y_2=\frac{1}{3a\sin\theta\cos^4\theta}.}$$

Let $(\bar{x},\bar{y})$ be the centre of curvature, then

$$\bar{x}=x-\frac{y_1(1+y_1^2)}{y_2}$$

$$=a\cos^3\theta-\frac{(-\tan\theta)(1+\tan^2\theta)}{\frac{1}{3a\sin\theta\cos^4\theta}}$$

$$=a\cos^3\theta+\frac{\sin\theta}{\cos\theta}\left(1+\frac{\sin^2\theta}{\tan^2\theta}\right)\cdot 3a\sin\theta\cos$$

$$= a\cos^3\theta + \frac{\sin\theta}{\cos\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right) 3a\sin\theta\cos^4\theta$$

$$\boxed{\bar{x} = a\cos^3\theta + 3a\sin^2\theta\cos\theta}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= a\sin^3\theta + \frac{1+\tan^2\theta}{\frac{1}{3a\sin\theta\cos^4\theta}}$$

$$= a\sin^3\theta + \left(1+\frac{\sin^2\theta}{\cos^2\theta}\right) 3a\sin\theta\cos^4\theta$$

$$= a\sin^3\theta + \left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right) 3a\sin\theta\cos^4\theta$$

$$\boxed{\bar{y} = a\sin^3\theta + 3a\sin\theta\cos^2\theta.}$$

$$\bar{x}+\bar{y} = a\cos^3\theta + 3a\sin^2\theta\cos\theta + a\sin^3\theta + 3a\sin\theta\cos^2\theta$$

$$= a[\cos^3\theta + 3\cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta + \sin^3\theta]$$

$$= a[\cos\theta + \sin\theta]^3$$

$$\bar{x}-\bar{y} = a\cos^3\theta + 3a\sin^2\theta\cos\theta - a\sin^3\theta - 3a\sin\theta\cos^2\theta$$

$$= a[\cos^3\theta - 3\cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta - \sin^3\theta]$$

$$= a(\cos\theta - \sin\theta)^3$$

$$(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = [a(\cos\theta+\sin\theta)^3]^{2/3} + [a(\cos\theta-\sin\theta)^3]^{2/3}$$

$$= a^{2/3}[(\cos\theta+\sin\theta)^2 + (\cos\theta-\sin\theta)^2]$$

$$= a^{2/3}[2(\cos^2\theta+\sin^2\theta)]$$

$$= 2a^{2/3}$$

$$\therefore (x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$$

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(5) find the evolute of the rectangular hyperbola

$xy = c^2$

Soln

The parametric eqn of $xy=c^2$ is

$x = ct \qquad y = \frac{c}{t}$

$\frac{dx}{dt} = c \qquad \frac{dy}{dt} = -\frac{c}{t^2}$

$y_1 = \frac{dy/dt}{dx/dt} = \frac{-c/t^2}{c}$

$$\boxed{y_1 = -\frac{1}{t^2}}$$

$y_2 = \frac{d}{dt}\left(-\frac{1}{t^2}\right)\frac{dt}{dx}$

$= \frac{2}{t^3} \cdot \frac{1}{c}$

$$\boxed{y_2 = \frac{2}{ct^3}}$$

let $(\bar{x}, \bar{y})$ be the center of curvature, then

$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$

$= ct - \frac{\left(-\frac{1}{t^2}\right)\left(1+\frac{1}{t^4}\right)}{\frac{2}{ct^3}}$

$= ct + \frac{ct}{2}\left(1+\frac{1}{t^4}\right)$

$= ct + \frac{ct}{2} + \frac{c}{2t^3}$

$$\boxed{\bar{x} = \frac{3ct}{2} + \frac{c}{2t^3}}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= \frac{c}{t} + \frac{1+\frac{1}{t^4}}{\frac{2}{ct^3}}$$

$$= \frac{c}{t} + \frac{ct^3}{2}\left(1+\frac{1}{t^4}\right) = \frac{c}{t} + \frac{ct^3}{2} + \frac{c}{2t}$$

$$\boxed{\bar{y} = \frac{3c}{2t} + \frac{ct^3}{2}}$$

$$\bar{x}+\bar{y} = \frac{3ct}{2} + \frac{c}{2t^3} + \frac{3c}{2t} + \frac{ct^3}{2}$$

$$= \frac{c}{2}\left(t^3+3t+\frac{3}{t}+\frac{1}{t^3}\right)$$

$$= \frac{c}{2}\left(t+\frac{1}{t}\right)^3$$

$$\bar{x}-\bar{y} = \frac{3ct}{2} + \frac{c}{2t^3} - \frac{3c}{2t} - \frac{ct^3}{2}$$

$$= -\frac{c}{2}\left(t^3-3t+\frac{3}{t}-\frac{1}{t^3}\right)$$

$$= -\frac{c}{2}\left(t-\frac{1}{t}\right)^3$$

$$(\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[\left(t+\frac{1}{t}\right)^2 - \left(t-\frac{1}{t}\right)^2\right]$$

$$= 4\left(\frac{c}{2}\right)^{2/3}$$

$$(x+y)^{2/3} + (x-y)^{2/3} = 4\left(\frac{c}{2}\right)^{2/3}$$

⑥ find the evolute of the cycloid $x = a(\theta + \sin\theta)$
$y = a(1 - \cos\theta)$

Soln

The parametric equation of cycloid is

$x = a(\theta + \sin\theta)$ $\quad$ $y = a(1 - \cos\theta)$

$\frac{dx}{d\theta} = a(1 + \cos\theta)$ $\quad$ $\frac{dy}{d\theta} = a\sin\theta$

$\sin 2\theta = 2\sin\theta\cos\theta.$
$\sin 2(\frac{\theta}{2}) = 2\sin\frac{\theta}{2}\cos\theta/2$
$\cos 2\theta = \cos^2\alpha - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$

$1 + \cos\theta = 2\cos^2\theta/2$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{2a\sin\theta/2\cos\theta/2}{2a\cos^2\theta/2}$$

$$\boxed{\frac{dy}{dx} = \tan\theta/2}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta/2)$$

$$= \frac{d}{d\theta}(\tan\theta/2)\frac{d\theta}{dx}$$

$$= \sec^2\theta/2 \cdot \frac{1}{2}\left(\frac{1}{a(1+\cos\theta)}\right)$$

$$= \frac{1}{2\cos^2\theta/2} \cdot \frac{1}{2a\cos^2\theta/2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{4a\cos^4\theta/2}}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a(\theta + \sin\theta) - \frac{(\tan\frac{\theta}{2})(1 + \tan^2\frac{\theta}{2})}{\frac{1}{4a\cos^4\theta/2}}$$

$$= a(\theta + \sin\theta) - 4a\left(\frac{\sin\theta/2}{\cos\theta/2}\right) \times (\sec^2\theta/2) \times \cos^4\theta/2$$

$$= a(\theta + \sin\theta) - 2a.2\sin\theta/2\cos\theta/2$$

$= a(\theta + \sin\theta) - 2a\sin\theta$

$$\boxed{\bar{x} = a(\theta - \sin\theta)}$$

$\bar{y} = y + \frac{1+y_1^2}{y_2}$

$= a(1-\cos\theta) + \frac{1+\tan^2\frac{\theta}{2}}{\frac{1}{4a\cos^4\theta/2}}$

$= a(1-\cos\theta) + 2a \cdot 2\cos^2\theta/2$

$= a(1-\cos\theta) + 2a(1+\cos\theta)$

$$\boxed{\bar{y} = 2a + a(1+\cos\theta)}$$

Evolute of the cycloid is another cycloid.

---

(7) Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$ is another cycloid

Soln

$x = a(\theta - \sin\theta) \qquad y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = a(1-\cos\theta) \qquad \frac{dy}{d\theta} = a(0+\sin\theta) = a\sin\theta.$

$$\boxed{1-\cos\theta = 2\sin^2\frac{\theta}{2}}$$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \cot\theta/2$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\cot\frac{\theta}{2}\right)$

$= \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\frac{d\theta}{dx}$

$= -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} \cdot \frac{d\theta}{dx}$

$= -\frac{1}{2}\frac{1}{\sin^2\theta/2} \cdot \frac{1}{a(1-\cos\theta/2}$

$= -\frac{1}{2} \cdot \frac{1}{\sin^2\theta/2} \cdot \frac{1}{2a\sin^2\theta/2}$

$$\frac{d^2y}{dx^2} = -\frac{1}{4a\sin^4\frac{\theta}{2}}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a(\theta - \sin\theta) + \frac{(1+\cot^2\theta/2)\cdot\cot\theta/2}{\frac{1}{4a\sin^4\theta/2}}$$

$$= a(\theta - \sin\theta) + 2a\sin\theta.$$

$$= a\theta - a\sin\theta + 2a\sin\theta.$$

$$= a\theta + 2a\sin\theta - a\sin\theta.$$

$$= a\theta + a\sin\theta$$

$$\boxed{\bar{x} = a(\theta + \sin\theta)}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= a(1-\cos\theta) - \frac{(1+\cot^2\theta/2)}{\frac{1}{4a\sin^4\theta/2}}$$

$$= a(1-\cos\theta) - \operatorname{cosec}^2\theta/2 \cdot 4a\sin^4\theta/2$$

$$= a(1-\cos\theta) - \frac{1}{\sin^2\theta/2}\cdot 4a\sin^4\theta/2$$

$$= a(1-\cos\theta) - 4a\sin^2\theta/2$$

$$= a(1-\cos\theta) - 4a\left(\frac{1-\cos\theta}{2}\right)$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta)$$

$$= a - a\cos\theta - 2a + 2a\cos\theta$$

$$= -a + a\cos\theta.$$

$$\boxed{\bar{y} = -a(1-\cos\theta)}$$

$$\therefore x = a(\theta + \sin\theta)\ ;\ y = -a(1-\cos\theta)$$

$\therefore$ This is also a cycloid.

$1+\cot^2\theta = \operatorname{cosec}^2\theta$

$\cancel{(1+\operatorname{cosec}^2\theta/2)}$

$\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{\cos\theta/2}{\sin\theta/2}.$

$\frac{1}{\sin^2\theta/2}\cdot\frac{\cos\theta/2}{\sin\theta/2}\cdot 4a\sin^4\frac{\theta}{2}$

$= \cos\frac{\theta}{2}\cdot 4a\sin\theta/2$

$= \frac{1}{2}4a\sin\theta$

$\sin 2\theta = 2\sin\theta\cos\theta.$

$\sin 2(\frac{\theta}{2}) = 2\sin\theta/2\cos\theta/2$

$\cancel{2\sin\theta/2} =$

$\frac{\sin\theta}{2} = \sin\theta/2\cos\theta/2$

$= 4a(\frac{1}{2}\sin\theta)$

$\sin^2\theta/2 = \frac{1-\cos\theta}{2}$

(8) Show that the evolute of the curve

$x = c\cos t + c\log\tan\frac{t}{2}$; $y = c\sin t$ is $y = c\cosh\frac{x}{c}$

Soln

$x = c\cos t + c\log\tan\frac{t}{2}$

$\frac{dx}{dt} = -c\sin t + \frac{c}{\tan t/2} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}$

$= -c\sin t + \frac{c\cos\frac{t}{2}}{\sin t/2} \cdot \frac{1}{2\cos^2 t/2}$

$= -c\sin t + \frac{c}{\sin t}$

$= \frac{c(1-\sin^2 t)}{\sin t}$

$\boxed{\frac{dx}{dt} = \frac{c\cos^2 t}{\sin t}}$

$y = c\sin t$

$\frac{dy}{dt} = c\cos t$

$\frac{dy}{dx} = \frac{c\cos t\sin t}{c\cos^2 t}$

$\boxed{\frac{dy}{dx} = \tan t}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t)$

$= \frac{d}{dt}(\tan t) \cdot \frac{dt}{dx}$

$= \sec^2 t \cdot \frac{dt}{dx}$

$= \frac{1}{\cos^2 t} \cdot \frac{\sin t}{c\cos^2 t} = \frac{\sin t}{c\cos^4 t}$

$\boxed{\frac{d^2y}{dx^2} = \frac{\sin t}{c\cos^4 t}}$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= c\cos t + c\log(\tan\tfrac{t}{2}) - \frac{\tan t\,(1+\tan^2 t)}{\frac{\sin t}{c\cos^4 t}}$$

$$= c\cos t + c\log(\tan\tfrac{t}{2}) - \frac{\tan t \cdot c\cos^4 t \cdot \sec^2 t}{\sin t}$$

$$= c\cos t + c\log(\tan\tfrac{t}{2}) - c\cos t$$

$$\boxed{\bar{x} = c\log\left(\tan\frac{t}{2}\right)} \text{ ——— } (1)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= c\sin t + \frac{c\cos^4 t}{\sin t}(1+\tan^2 t)$$

$$= c\sin t + \frac{c\cos^2 t}{\sin t}$$

$$\boxed{\bar{y} = \frac{c}{\sin t}} \text{ ——— } (2)$$

The locus of $(\bar{x}, \bar{y})$ is

$$x = c\log\tan\frac{t}{2}$$

$$y = \frac{c}{\sin t}$$

$$\frac{y}{c} = \frac{1}{\sin t}.$$

$$= \frac{1+\tan^2\frac{t}{2}}{2\tan\theta/2}$$

$$\frac{y}{c} = \frac{1}{2}\left[\frac{1}{\tan\frac{\theta}{2}} + \tan\frac{\theta}{2}\right]$$

$$\boxed{\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}}$$

$\frac{x}{c} = \log\left(\tan\frac{t}{2}\right)$

$e^{\frac{x}{c}} = \tan\frac{t}{2}$

$\therefore \frac{y}{c} = \frac{1}{2}\left(\frac{1}{e^{x/c}} + e^{x/c}\right)$

$\frac{y}{c} = \frac{1}{2}\left(e^{-\frac{x}{c}} + e^{x/c}\right)$

$\frac{y}{c} = \cosh\frac{x}{c}$

$\boxed{y = c\cosh\frac{x}{c}}$

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Radius of Curvature in the polar Co-ordinates

Formula

$$\boxed{\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}}$$

Let $r = f(\theta)$ be the given curve in polar Co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ may be regarded as the parametric equations of the given curve the parameter being θ.

$$\boxed{\rho = \frac{\left\{r^2 + r_1^2\right\}^{3/2}}{r^2 + 2r_1^2 - r r_2}}$$

① Find the radius of curvature of the cardioid $r = a(1-\cos\theta)$

Soln

Given $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} = \left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}$$

$$= \left[a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta\right]^{3/2}$$

$$= \left[a^2 + a^2(\cos^2\theta + \sin^2\theta) - 2a\cos\theta\right]^{3/2}$$

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$$= \left[2a^2 - 2a^2\cos\theta\right]^{3/2}$$

$$\boxed{1 - \cos\theta = 2\sin^2\theta/2}$$

$$= \left[2a^2(1-\cos\theta)\right]^{3/2}$$

$$= \left[2a^2 . 2\sin^2\theta/2\right]^{3/2}$$

$$= \left[4a^2 . \sin^2\theta/2\right]^{3/2}$$

$$= 2^3 a^3 \sin^3\theta/2$$

$$\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} = 8a^3\sin^3\theta/2$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)$$

$$= a^2(1+\cos^2\theta - 2\cos\theta) + 2a^2\sin^2\theta$$
$$- a^2\cos\theta + a^2\cos^2\theta.$$

$$= a^2 + a^2\cos^2\theta - 2a^2\cos\theta + 2a^2\sin^2\theta$$
$$- a^2\cos\theta + \underline{a^2\cos^2\theta}.$$

$$= a^2 + 2a^2(\cos^2\theta + \sin^2\theta) - 3a^2\cos\theta$$

$$= 3a^2 - 3a^2\cos\theta.$$

$$= 3a^2(1-\cos\theta).$$

$$= 3a^2 . 2\sin^2\theta/2$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = 6a^2\sin^2\theta/2$$

$$\rho = \frac{\{r^2 + r_1^2\}^{3/2}}{r^2 + 2r_1 - rr_2}$$

$$= \frac{8a^3\sin^3\theta/2}{6a^2\sin^2\theta/2}$$

$$= \frac{4}{3} a \sin \theta/2$$

---

(2) Find the radius of curvature of the curve $r^n = a^n \cos n\theta$ at any point (r, θ). Hence prove that the radius of curvature of the curve $r^2 = a^2 \cos 2\theta$ is $\frac{a^2}{3r}$.

Soln

Given $r^n = a^n \cos n\theta$

Taking log on both sides

$$\log r^n = \log(a^n \cos n\theta)$$

$$n \log r = n \log a + \log \cos n\theta$$

D.w.r.to θ

$$n \cdot \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-\sin n\theta) \cdot n$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$$\frac{dr}{d\theta} = -r \tan n\theta$$

$$\frac{d^2r}{d\theta^2} = -r\left[\sec^2 n\theta \,(n)\right] - \tan n\theta \,(r')$$

$$= -nr\sec^2 n\theta - r' \tan n\theta$$

$$= -nr\sec^2 n\theta - (-r\tan n\theta)\tan n\theta$$

$$r'' = -nr\sec^2 n\theta + r\tan^2 n\theta$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{[r^2 + (-r\tan n\theta)^2]^{3/2}}{r^2 - r(-nr\sec^2 n\theta + r\tan^2 n\theta) + 2(-r\tan n\theta)^2}$$

$$= \frac{(r^2)^{3/2}[1+\tan^2 n\theta]^{3/2}}{r^2 + nr^2\sec^2 n\theta - r^2\tan^2 n\theta + 2r^2\tan^2 n\theta}$$

$$= \frac{r^3(\sec^2 n\theta)^{3/2}}{r^2 + nr^2\sec^2 n\theta + r^2\tan^2 n\theta}$$

$$= \frac{r^3(\sec^2 n\theta)^{3/2}}{r^2(1+\tan^2 n\theta) + nr^2\sec^2 n\theta}$$

$$= \frac{r^3(\sec^2 n\theta)^{3/2}}{r^2\sec^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta(1+n)}$$

$$= \frac{r\sec n\theta}{n+1}$$

$$= \frac{r}{n+1}\cdot\frac{1}{\cos n\theta}$$

$$= \frac{r}{n+1}\left(\frac{1}{\frac{r^n}{a^n}}\right)$$

$r^n = a^n\cos n\theta$

$\cos n\theta = \frac{r^n}{a^n}$

$$\rho = \frac{a^n}{(n+1)r^{n-1}}$$

Put $n=2$

$$\boxed{\rho = \frac{a^2}{3r}}$$

(3) show that the radius of curvature of the curve $r^2 = a^2 \cos 2\theta$ is $\frac{a^2}{3r}$

Soln

$$r^2 = a^2 \cos 2\theta$$

D.w.r.to θ

$$2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta \quad — (1)$$

$$r \frac{dr}{d\theta} = -a^2 \sin 2\theta$$

$$r \frac{d^2 r}{d\theta^2} + \left(\frac{dr}{d\theta}\right)^2 = -2a^2 \sin 2\theta.$$

$$= -2r^2$$

$$\therefore r r_2 = -2r^2 - r_1^2$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$$= \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 + 2r^2 + r_1^2}$$

$$= \frac{(r^2 + r_1^2)^{3/2}}{3(r^2 + r_1^2)}$$

$$= \frac{1}{3}(r^2 + r_1^2)^{1/2}$$

$$= \frac{1}{3}\left[r^2 + \frac{a^4 \sin 2\theta}{r^2}\right]^{1/2}$$

$$= \frac{1}{3}\left[\frac{(r^4 + a^4 \sin^2 2\theta)^{1/2}}{r}\right]$$

$$= \frac{1}{3r}\left(a^4 \cos^2 2\theta + a^4 \sin^2 2\theta\right)^{1/2}$$

$$\boxed{\rho = \frac{a^2}{3r}}$$

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④ Show that in the cardioid $r = a(1+\cos\theta)$, $\frac{\rho^2}{r}$ is constant.

Soln

Given $r = a(1+\cos\theta)$

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{\{[a(1+\cos\theta)]^2 + (-a\sin\theta)^2\}^{3/2}}{[a(1+\cos\theta)]^2 + 2(-a\sin\theta)^2 - a(1+\cos\theta)(-a\cos\theta)}$$

$$= \frac{[a^2(1+2\cos\theta+\cos^2\theta+\sin^2\theta)]^{3/2}}{a^2[1+2\cos\theta+\cos^2\theta+2\sin^2\theta+\cos^2\theta+\cos\theta]}$$

$$= \frac{[2a^2(1+\cos\theta)]^{3/2}}{3a^2(1+\cos\theta)}$$

$$= \frac{2\sqrt{2}\cdot a^3(1+\cos\theta)^{3/2}}{3a^2(1+\cos\theta)}$$

$$= \frac{2\sqrt{2}\cdot a(1+\cos\theta)^{1/2}}{3}$$

$$= \frac{2\sqrt{2}\cdot a}{3}\, 2\cos^2\theta/2$$

$$\rho = \frac{4a}{3}\cos\theta/2$$

The radius of curvature in term of the variable r is given by

$$\rho = \frac{[2a^2(1+\cos\theta)]^{3/2}}{3a^2(1+\cos\theta)}$$

$$= \frac{[2a(a(1+\cos\theta))]^{3/2}}{3a(a(1+\cos\theta)}$$

$$= \frac{(2ar)^{3/2}}{3ar}$$

$$= \frac{2}{3}(2ar)^{1/2}$$

Therefore $\frac{\rho^2}{r} = \frac{8a}{9}$ is a constant

---

(5) Show that the curvature of the curve $r = a\theta$ and $r\theta = a$ at their point of intersection are in the ratio 3:1

Soln

The given curves are

$$r = a\theta \quad \text{——} (1)$$

$$r\theta = a \quad \text{——} (2)$$

Solving equation (1) and (2), $r = a\theta$

$r = \frac{a}{\theta}$

$$a\theta = \frac{a}{\theta}$$

$$\theta^2 = 1$$

$$\theta = \pm 1$$

$\therefore \theta = \pm 1$ are the points of intersection of the two curves.

From the given curve

$$r = a\theta$$

$$r_1 = a$$

$$r_2 = 0$$

$$= \frac{[2a(a(1+\cos\theta))]^{3/2}}{3a(a(1+\cos\theta)}$$

$$= \frac{(2ar)^{3/2}}{3ar}$$

$$= \frac{2}{3}(2ar)^{1/2}$$

There fore $\frac{\rho^2}{r} = \frac{8a}{9}$ is a constant

---

⑤ Show that the curvature of the curve $r = a\theta$ and $r\theta = a$ at their point of intersection are in the ratio $3:1$

Soln

The given curves are

$$r = a\theta \quad \text{—— ①}$$

$$r\theta = a \quad \text{—— ②}$$

Solving equation ① and ②, $r = a\theta$

$r = \frac{a}{\theta}$

$$a\theta = \frac{a}{\theta}$$

$$\theta^2 = 1$$

$$\theta = \pm 1$$

$\therefore \theta = \pm 1$ are the points of intersection of the two curves.

From the given curve

$$r = a\theta$$

$$r_1 = a$$

$$r_2 = 0$$

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$$\rho = \frac{\{r^2 + r_1^2\}^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{\{a^2\theta^2 + a^2\}^{3/2}}{a^2\theta^2 + 2a^2}$$

$$= \frac{a(1+\theta^2)^{3/2}}{2+\theta^2}$$

$$(\rho)_{\theta=\pm 1} = \frac{a\,2\sqrt{2}}{3}$$

from the eqn ②

$$r\theta = a$$

$$r = \frac{a}{\theta}$$

$$r_1 = \frac{-a}{\theta^2}$$

$$r_2 = \frac{2a}{\theta^3}$$

$$\rho = \frac{\{r^2 + r_1^2\}^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{\left(\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}\right)^{3/2}}{\frac{a^2}{\theta^2} + \frac{2a^2}{\theta^4} - \frac{2a}{\theta^3}\cdot\frac{a}{\theta}}$$

$$= \frac{a(1+\theta^2)^{3/2}}{\theta^4}$$

$$(\rho)_{\theta=\pm 1} = 2\sqrt{2}\,a$$

Thus the curvature of the two curves at their point of intersection are $\frac{3}{2\sqrt{2}a}$ and $\frac{1}{2\sqrt{2}a}$ which are in the ratio $3:1$

(6) Find the radius of curvature of $r = ke^{\theta \cot\alpha}$ where k and α are constant.

Soln

Given $r = ae^{\theta \cot\alpha}$

$$r_1 = a\cot\alpha \cdot e^{\theta\cot\alpha}$$

$$r_2 = a(\cot\alpha)^2 e^{\theta\cot\alpha}$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{\left[a^2 e^{2\theta\cot\alpha}(1+\cot^2\alpha)\right]^{3/2}}{(ae^{\theta\cot\alpha})^2+(ae^{\theta\cot\alpha})^2\, 2\cot^2\alpha-(ae^{\theta\cot\alpha})^2\cot^2\alpha}$$

$$= \frac{\left[a^2 e^{2\theta\cot\alpha}(1+\cot^2\alpha)\right]^{3/2}}{(ae^{\theta\cot\alpha})^2(1+2\cot^2\alpha-\cot^2\alpha)}$$

$$= \frac{a^3 e^{3\theta\cot\alpha}(\operatorname{cosec}^2\alpha)^{3/2}}{a^2 e^{2\theta\cot\alpha}(\operatorname{cosec}^2\alpha)}$$

$$= ae^{\theta\cot\alpha}\cdot\operatorname{cosec}\alpha$$

$$\rho = r\operatorname{cosec}\alpha.$$

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P-r equation; Pedal equation of curve

Formula

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

① Prove that the P-r equation of the cardioid $r = a(1-\cos\theta)$ is $p^2 = \frac{r^2}{2a}$

Soln

$$r = a(1-\cos\theta)$$

$$\frac{dr}{d\theta} = a\sin\theta.$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}a^2\sin^2\theta$$

$$= \frac{r^2 + a^2\sin^2\theta}{r^4}$$

$$= \frac{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}{r^4}$$

$$= \frac{a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta}{r^4}$$

$$= \frac{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta}{r^4}$$

$$= \frac{a^2 + a^2(\cos^2\theta + \sin^2\theta) - 2a^2\cos\theta}{r^4}$$

$$= \frac{2a^2 - 2a^2\cos\theta}{r^4}$$

$$= \frac{2a^2(1-\cos\theta)}{r^4}$$

$$= \frac{2a^2}{r^4}\cdot r$$

$$\frac{1}{p^2} = \frac{2a}{r^3}$$

$$\boxed{p^2 = \frac{r^3}{2a}}$$

(2) From the polar equation of the parabola, show that $p^2 = ar$

Soln

polar equation of the parabola is $\frac{2a}{r} = 1 - \cos\theta$

Diff w.r. to θ.

$$-\frac{2a}{r^2}\frac{dr}{d\theta} = \sin\theta$$

$$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{\sin\theta}{2a}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{\sin^2\theta}{4a^2}$$

$\frac{2a}{r} = 1 - \cos\theta.$

$\frac{1}{r} = \frac{1-\cos\theta}{2a}.$

$$= \frac{(1-\cos\theta)^2}{4a^2} + \frac{\sin^2\theta}{4a^2}$$

$$= \frac{1 + \cos^2\theta - 2\cos\theta + \sin^2\theta}{4a^2}$$

$$= \frac{2 - 2\cos\theta}{4a^2}$$

$$= \frac{2(1-\cos\theta)}{4a^2}$$

$$= \frac{2}{4a^2} \cdot \frac{2a}{r}$$

$$\frac{1}{p^2} = \frac{1}{ar}$$

$$\boxed{p^2 = ar}$$

③ find the (P-r) equation for the curve $r\sin\theta + a = 0$

Soln

Given $r\sin\theta + a = 0$

$$\sin\theta \frac{dr}{d\theta} + r \cos\theta = 0$$

$$\therefore \frac{dr}{d\theta} = -r\cot\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} r^2 \cot^2\theta$$

$$= \frac{1+\cot^2\theta}{r^2}$$

$$= \frac{\operatorname{cosec}^2\theta}{r^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{1}{\sin^2\theta}.$$

$r\sin\theta + a = 0$

$r\sin\theta = -a$

$\sin\theta = \frac{-a}{r}$

$$= \frac{1}{r^2}\left(\frac{r^2}{a^2}\right)$$

$$\frac{1}{p^2} = \frac{1}{a^2}$$

$$\boxed{p^2 = a^2}$$

Note

The radius of curvature

$$\boxed{\rho = r\frac{dr}{dp}}$$

④ Find the radius of curvature of the curve $r^2 = a^2 \sin 2\theta$

<u>soln</u>

Given $r^2 = a^2 \sin 2\theta$

$$2r\frac{dr}{d\theta} = 2a^2 \cos 2\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\,\frac{a^4 \cos^2 2\theta}{r^2}$$

$$= \frac{r^4 + a^4 \cos^2 2\theta}{r^6}$$

$$= \frac{a^4 \sin^2 2\theta + a^4 \cos^2 2\theta}{r^6}$$

$$\frac{1}{p^2} = \frac{a^4}{r^6}$$

$$p^2 = \frac{r^6}{a^4}$$

$$\boxed{p = \frac{r^3}{a^2}}$$

Diff w. r. to r

$$\frac{dp}{dr} = \frac{3r^2}{a^2}$$

$$\therefore \rho = r\frac{dr}{dp}$$

$$= r\left(\frac{a^2}{3r^2}\right)$$

$$\boxed{\rho = \frac{a^2}{3r}}$$

Unit-IV

Linear Asymptotes

Defn If a straight line cuts a curve in two points at an infinite distance from the origin itself not lying wholly at infinity, it is called as asymptotes to the curve.

Rules

In the highest degree terms put $x=1$ and $y=m$ this gives $\phi_n(m)=0$ hence m is found

Form $\phi_{n-1}(m)$ in a similar way from terms of degree $n-1$ and differentiate $\phi_{n-1}(m)$ then the values of c are got from the formula $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$ by putting $m=m_1, \dots m_n$

Cor

As $\phi_n(m)=0$ is of the n^{th} degree, there are n values of m: hence there exist n asymptote real or imaginary for a curve of n^{th} degree.

Cor If the degree of an equation be odd, there exists atleast one real root as imaginary roots occur in pairs only. Hence no curve of an odd degree can be closed. A curve of odd degree cannot have an even number of real asymptotes.

Cor 3

If the term y^n be absent in the equation of the curve, then the term m^n will be missing in $\phi_n(m)=0$. Hence the degree of this equation is apparently reduced by one. But we know that one value of m is ∞. Therefore the corresponding asymptotes is perpendicular to the axis of x. ie parallel to the y-axis.

Asymptotes parallel to the axis.

If the eqn of the curve be arranged in the form $a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\dots+a_ny^n$

$+b_1x^{n-1}+b_2x^{n-2}y+\dots+b_ny^{n-1}+c_2x^{n-2}+\dots$

Rearranging in descending powers of x, we

$$a_0x^n+(a_1y+b_1)x^{n-1}+\dots+=0$$

Ex

① Find the asymptotes of the cubic
$$y^3-6xy^2+11x^2y-6x^3+x+y=0$$

soln

Given $y^3-6xy^2+11x^2y-6x^3+x+y=0$

The highest degree terms ie, the third degree terms are $y^3-6xy^2+11x^2y-6x^3$

put $x=1$ $y=m$

we get

$$\phi_3(m)=m^3-6m^2+11m-6=0$$

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$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$m=1$
$m=2$
$m=3$

$m^2-5m+6=0$
$(m-3)(m-2)=0$
$m=2,3$

$\phi_2(m)=0$ as there are no second degree terms.

Hence $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = 0$

$\therefore$ The three asymptotes

$y=x,\ y=2x,\ y=3x.$

---

(2) Find the asymptotes of $x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0$

<u>soln</u>

Given $x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0$

Here the highest degree terms are

$x^3+2x^2y-xy^2-2y^3$

putting $x=1$ and $y=m$

$\phi_3(m) = 1+2m-m^2-2m^3 = 0$

$\boxed{\begin{aligned}\phi_3'(m) &= 2-2m-6m^2 \\ &= 2(1-m-3m^2)\end{aligned}}$

$\cancel{\phi_3'(m) = 6m^2+2m-2 = 2(3m^2+m-1}$

$\phi_3(m) = 2m^3+m^2-2m-1=0$

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -2 & -1 \\ & & 2 & 3 & 1 \\ \hline & 2 & 3 & 1 & \boxed{0} \end{array}$$

$m=1,-1,-\frac{1}{2}$

$2m^2+3m+1=0$
$2m^2+2m+m+1=0$
$2m(m+1)+(m+1)=0$
$(m+1)(2m+1)=0$
$m=-1 \quad 2m+1=0$
$2m=-1$
$m=-\frac{1}{2}$

$\phi_2(m) = 4m^2+2m$

$c = -\dfrac{\phi_2(m)}{\phi_3'(m)}$

$= \dfrac{4m^2+2m}{2(1-m-3m^2)}$

$$= \frac{2m(2m+1)}{2(1-m-3m^2)}$$

$$= \frac{m(2m+1)}{3m^2+m-1}$$

For $m=1$

$$c = \frac{3}{3} = 1$$

$c=1$

The corresponding asymptote is $y = x+1$

For $m=-1$

$$c = \frac{(-1)(-1)}{3-1-1} = \frac{1}{1} = 1$$

$c=1$

The corresponding asymptotes is $y=-x+1$

For $m = -1/2$

$$c = \frac{(-1/2)(2(-1/2)+1)}{3(-\frac{1}{2})^2+(-1/2)-1}$$

$$= \frac{(-1/2)(-1+1)}{3(\frac{1}{4})-1/2+1}$$

$c=0$

The corresponding asymptotes is $y = -\frac{x}{2}$

③ Find the rectilinear asymptotes of the curve

$$y^2(x^2-y^2)-2ay^3+2a^3x=0$$

Soln

Given $y^2(x^2-y^2)-2ay^3+2a^3x=0$

$\phi_4(m)=m^2(1-m^2)=0$

$m^2(1-m^2)=0$

$m^2=0 \quad 1-m^2=0$

$m^2=1$

$m=\pm 1$

$m=0,1,-1$

$$C=-\frac{\phi_3(m)}{\phi_4'(m)}$$

m^2-m^4

$2m-4m^3$

$2m(1-2m^2)$

$$=\frac{2am^3}{2m(1-2m^2)}$$

$$=\frac{am^2}{1-2m^2}$$

For $m=0$, $c=0$ the corresponding asymptotes is $\boxed{y=0}$

For $m=1$, $c=-a$ the corresponding asymptote is $\boxed{y=x-a}$

For $m=-1$, $c=-a$ the corresponding asymptotes $\boxed{y=-x-a}$

---

④ Find the asymptotes of $x^3+y^3-3axy=0$

Soln

Given $x^3+y^3-3axy=0$

$\phi_3(m)=1+m^3=0$

$\phi_2(m)=-3am$

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$\varphi_3(m) = 1+m^3 = 0$

$m = -1$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \lfloor 0 \end{array}$$

$m^2 - m + 1 = 0$

$\frac{1 \pm \sqrt{1-4}}{2}$

$\frac{1 \pm \sqrt{3}}{2}$

$c = \frac{-\varphi_2(m)}{\varphi_3'(m)}$

$= \frac{-3am}{3m^2} = \frac{a}{m}$

put $m = -1$

$c = -a$

$\therefore$ The Corresponding asymptote is

$y = -x - a.$

$\therefore \boxed{x + y + a = 0}$

Special cases

Let

$$x^n \phi_n(m) + x^{n-1}\left[c\phi_n'(m) + \phi_{n-1}(m)\right]$$

$$+ x^{n-2}\left[\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m)\right] + \dots = 0 \quad \text{--- } (*)$$

Let us consider the equation $\phi_n(m) = 0$ --- (1)

and $c\phi_n'(m) + \phi_{n-1}(m) = 0$ --- (2) of (*)

Case 1

Suppose the roots $m_1, m_2 \dots m_n$ of $\phi_n(m) = 0$ are all different so that $\phi_n'(m) \neq 0$ for any of these roots.

Let $\phi_n(m) = 0$ and $\phi_{n-1}(m) = 0$ have the common factor, any $m - m_1$; then $c_1 = 0$. The corresponding asymptotes is $y = m_1 x$ passing through the origin

Case 2

Suppose two roots m_1 and m_2 of $\phi_n(m) = 0$ are equal then $\phi_n'(m_1) = 0$ if $\phi_{n-1}(m_1) \neq 0$ is infinite.

c as determined from (2) is infinite.

The line $y = m_1 x + c_1$ meets the curve in two points at infinity and makes an infinite intercept along the y-axis. Hence it lies wholly at infinity.

Case 3

$\varphi_n(m)=0$ have two ~~roots~~ equal roots, each equal to m_1 say; then $\varphi_n'(m_1)=0$. If m_1 also satisfies $\varphi_{n-1}(m_1)=0$, C cannot be determined from ②. So we resort to the coefficient of x^{n-2} in ② of ④ and make it satisfy

$$\frac{c^2}{2}\varphi_n''(m)+c\varphi_{n-1}'(m)+\varphi_{n-2}(m)=0 \quad \text{——— I}$$

Let the two roots of this equation be c_1, c_2 real (or) imaginary. Then the two corresponding asymptotes $y=m_1x+c_1$ and $y=m_1x+c_2$ are parallel so that each cuts the curve in three points of infinity.

Example:

Find the asymptotes of

$$x^3+2x^2y-4xy^2-8y^3-4x+8y=1$$

soln

Given $x^3+2x^2y-4xy^2-8y^3-4x+8y=1$.

$\varphi_3(x)=x^3+2x^2y-4xy^2-8y^3$

~~$\varphi_n(m)=1+$~~

$\varphi_n(m)=1+2m-4m^2-8m^3=0$

$(1-2m)(1+2m)^2=0$

$m_1=-\frac{1}{2}$

$m_2=-\frac{1}{2}$

$m_3=\frac{1}{2}$

$-\frac{1}{2}$	8	4	-2	-1
		-4	0	1
	8	0	-2	0

$8m^2-2=0$

$4m^2-1=0$

4

$\varphi_n(m)=1+2m-4m^2-8m^3$

$\varphi'=2-8m-24m^2$

$\varphi''=-8-48m$

c is determined by $c\varphi_n'(m) + \varphi_{n-1}(m) = 0$

$$c[\cancel{}\, 2 - 8m - 24m^2] = 0 \quad \text{as } \varphi_{n-1}(m) = 0 \quad -① $$

For $m_3 = \frac{1}{2}$

$$c[2-4-6] = -8c = 0$$

$$\boxed{c = 0}$$

corresponding asymptotes is $y = \frac{1}{2}x$.

$-4 + 8m$

$4(2m-1)$

For $m_1 = m_2 = -\frac{1}{2}$

① becomes $c(0) = 0$.

$\therefore$ c cannot be determined

we go to $\frac{c^2}{2}\varphi_n''(m) + c\varphi_{n-1}'(m) + \varphi_{n-2}(m) = 0$

$$\frac{c^2}{2}(-8-48m) + 0 + 4(2m-1) = 0$$

~~putting m = -1/2~~ $-4c^2(1+6m) + 4(2m-1) = 0$

putting $m = -\frac{1}{2}$ $\quad -4c^2(1+6(-\frac{1}{2})) + 4(2(-\frac{1}{2})-1) = 0$

$$-4c^2(1-3) + 4(-1-1) = 0$$

$$-4c^2(-2) + 4(-2) = 0$$

$$8c^2 - 8 = 0$$

$$c^2 - 1 = 0$$

$$c^2 = 1$$

$$\boxed{c = \pm 1}$$

The two parallal asymptotes are

$$y = -\frac{x}{2} + 1 \; ; \; y = -\frac{x}{2} - 1$$

~~$2y + x = 2$~~

$$\boxed{2y + x = 2} \qquad \boxed{2y + x = -2}$$

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② Find the asymptotes of $x^3+2x^2y+xy^2-x^2-xy+2=0$

Soln

Given $(x^3+2x^2y+xy^2)-(x^2+xy)+2=0$

The coefficients of highest powers of x and y are constants.

$\therefore$

$\varphi_3(m) = m^2+2m+1$

$\varphi_3'(m) = 2m+2$

$\varphi_3''(m) = 2$

$\varphi_2(m) = -(1+m)$

$\varphi_2'(m) = -1$

Put $x=1$, $y=m$

$\varphi_3(m) = 1+2m+m^2$

$\varphi_2(m) = -(1+m)$

$\varphi_3(m)=0 \Rightarrow m^2+2m+1=0$

$(m+1)(m+1)=0$

$m=-1,\ m=-1$

$\therefore$ $m=-1$ is a repeated root of $\varphi_3(m)=0$

Therefore there are two parallel asymptotes with slope $m=-1$

$\frac{c^2}{2}\varphi_3''(m) + c\,\varphi_2'(m) + \varphi_1(m) = 0$

$\varphi_3''(m)=2$; $\varphi_2'(m)=-1$; $\varphi_1(m)=0$

$\frac{c^2}{2}\cdot 2 + c(-1) = 0 \Rightarrow c^2-c=0$

$c(c-1)=0$

$c=0$

$c=1$

$\therefore$ The parallel asymptotes are

$y=-x$

$y=-x+1$

Another Method for finding asymptotes

Suppose the equation of curve of the n^{th} degree is put in the form $(ax+by+c)P_{n-1}+F_{n-1}=0$ where P_{n-1} and F_{n-1} denotes the polynomials in x and y. of the $(n-1)^{th}$ degree. Any straight line parallel to $ax+by+c=0$ cuts the curve in one point at infinity. To find the asymptotes, we seek that member of this family parallel to $ax+by+c=0$ which meets the curve in a second point at infinity. To find this, we allow x and y to tend to ∞ in the asymptotes direction $ax+by+c=0$ i.e $x:y:-b:a$.

$\therefore$ The asymptotes is

$$ax+by+c+\lim_{y=-\frac{a}{b}x\to\infty}\left(\frac{F_{n-1}}{P_{(n-1)}}\right)=0$$

If this limit is finite, the asymptotes we seek is found.

---

① Example.

Find the asymptotes of $x^3+y^3=3axy$

soln

Given $x^3+y^3=3axy$

$$x^3+y^3-3axy=0$$

This equation can be written as

$$(x+y)(x^2-xy+y^2)-3axy=0$$

Hence the asymptotes direction is $x+y=0$

Hence the asymptotes is

$$x+y+\lim_{y=-x\to\infty}\left(\frac{-3axy}{x^2-xy+y^2}\right)=0$$

ie $$x+y+\lim_{x\to\infty}\frac{3ax^2}{3x^2}=0$$

$$\boxed{x+y+a=0}$$

② Find the rectilinear asymptotes of

$$2x^4-5x^2y^2+3y^4+4x^3-6y^3+x^2+y^2-2xy+1=0$$

Soln

Given

$$2x^4-5x^2y^2+3y^4+4x^3-6y^3+x^2+y^2-2xy+1=0$$

~~Factorising the~~

Factorising the fourth degree terms

$$(2x^2-3y^2)(x^2-y^2)+4x^3-6y^3+x^2+y^2-2xy+1=0$$

$$4\left(\sqrt{\tfrac{3}{2}}\,y\right)^3-6y^3+\left(\sqrt{\tfrac{3}{2}}\,y\right)^2+y^2-2\left(\sqrt{\tfrac{3}{2}}\,y\right)+1$$

$$4\,\frac{3}{2}\sqrt{\frac{3}{2}}\,y^3-6y^3+\frac{3}{2}y^2+y^2-\sqrt{6}\,y+1$$

$$\left(\cancel{\frac{3\sqrt{3}}{\sqrt{2}}y^3-6y^3}\right)+\frac{5}{2}y^2-\sqrt{6}y+1$$

$3\sqrt{6}y^3-6y^3$

$$2x^2-3y^2=0$$
$$2x^2=3y^2$$
$$\sqrt{2}x=\pm\sqrt{3}y$$

$$x^2-y^2=0$$
$$x^2=y^2$$
$$x=\pm y$$

$x=\frac{\sqrt{3}}{\sqrt{2}}y$

Hence

$$(\sqrt{2}x-\sqrt{3}y)+\lim_{\sqrt{2}x=\sqrt{3}y\to\infty}\frac{4x^3-6y^3+x^2+y^2-2xy+1}{(\sqrt{2}x+\sqrt{3}y)(x^2-y^2)}=0$$

$$\sqrt{2}x-\sqrt{3}y+\lim_{y\to\infty}\frac{3\sqrt{6}\,y^3-6y^3+\frac{5}{2}y^2-\sqrt{6}\,y+1}{\cancel{2}\sqrt{3}\,y\cdot\frac{1}{\cancel{2}}y^2}=0$$

∴ One asymptotes is

$$\sqrt{2}x-\sqrt{3}y+(3\sqrt{2}-2\sqrt{3})=0 \qquad \text{——— (1)}$$

$$\frac{\cancel{y^3}(3\sqrt6-6)+\frac{5}{24}-\frac{\sqrt6}{y^2}+1)}{\cancel{y^3}(\sqrt3)}$$

$3\sqrt6-6+1=3\sqrt6-5$

Similarly

$$\sqrt2x+\sqrt3y+\lim_{\sqrt2x=-\sqrt3y\to\infty}\frac{4x^3-6y^3+x^2+y^2-2xy+1}{(\sqrt2x-\sqrt3y)(x^2-y^2)}=0$$

ie $$\sqrt2x+\sqrt3y+\lim_{y\to\infty}\frac{-3\sqrt6y^3-6y^3+5/2\,y^2+\sqrt6y^3+1}{2\sqrt3y\cdot\frac12y^2}=0$$

ie $\boxed{\sqrt2x+\sqrt3y+3\sqrt2+2\sqrt3=0}$ is second asymptote.

$$x-y+\lim_{y=x\to\infty}\frac{4x^3-6y^3+x^2+y^2-2xy+1}{(2x^2-3y^2)(x+y)}=0$$

ie $$x-y+\lim_{x\to\infty}\frac{-2x^3}{-x^2\cdot2x}=0$$

ie $\boxed{x-y+1=0}$ is third asymptote.

$$x+y+\lim_{y=-x\to\infty}\frac{4x^3-6y^3+y^2-2xy+1}{(2x^2-3y^2)(x-y)}=0$$

$$x+y+\lim_{x\to\infty}\frac{10x^3}{-x^2\cdot2x}=0$$

ie $\boxed{x+y-5}$ is fourth asymptote.

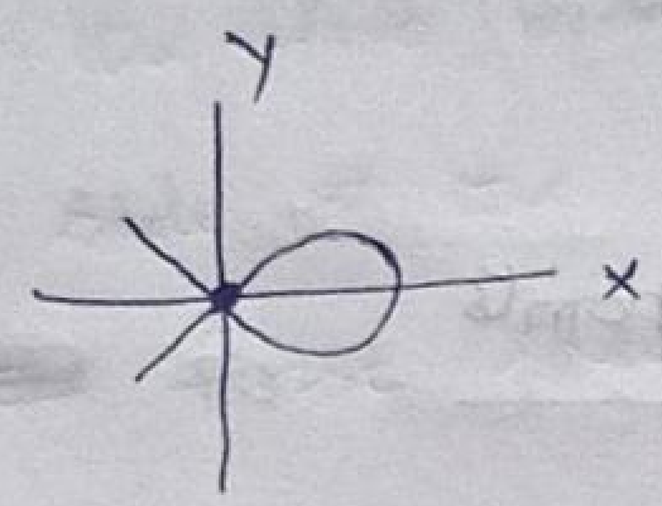
Singular points

Defn

Points through which more than one branch of a curve pass are called multiple points on the curve.

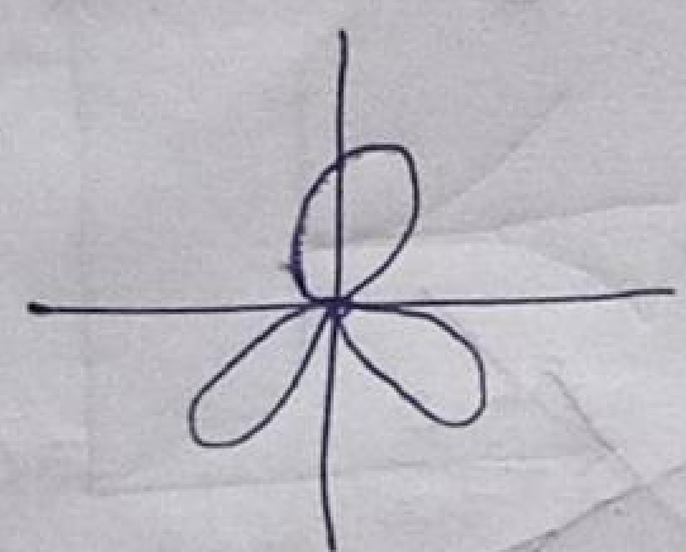
If two branches pass through a point is called a double point

Ex



If three branches of a curve pass through a point, that point is called a triple points

Ex



Generally if r branches of a curve pass through a point, that point is called multiple point of the ~~curve~~ r^{th} order on the curve.

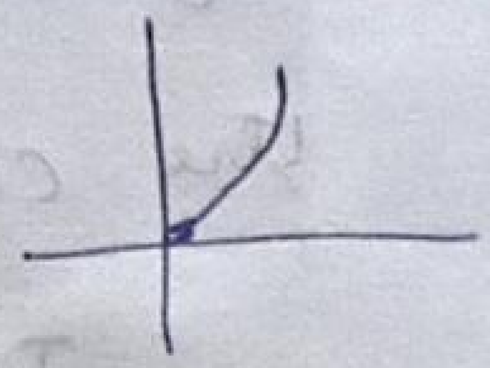
Defn

The points on the curve on which the curve behaves in an unusual way are called singular point

Singular points can be divided in to two groups

1. Points of inflexion:

These are points on one side of which the curve is convex and on the other side concave.



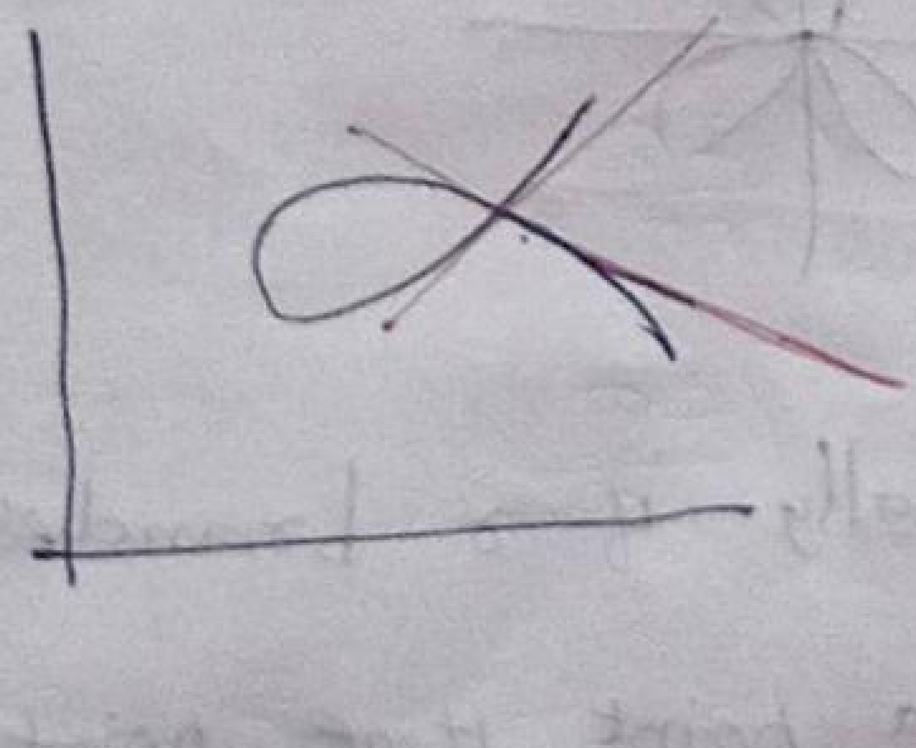
2. Multiple points

These are points through which more than one branch of the curve passes.

Types of douple points

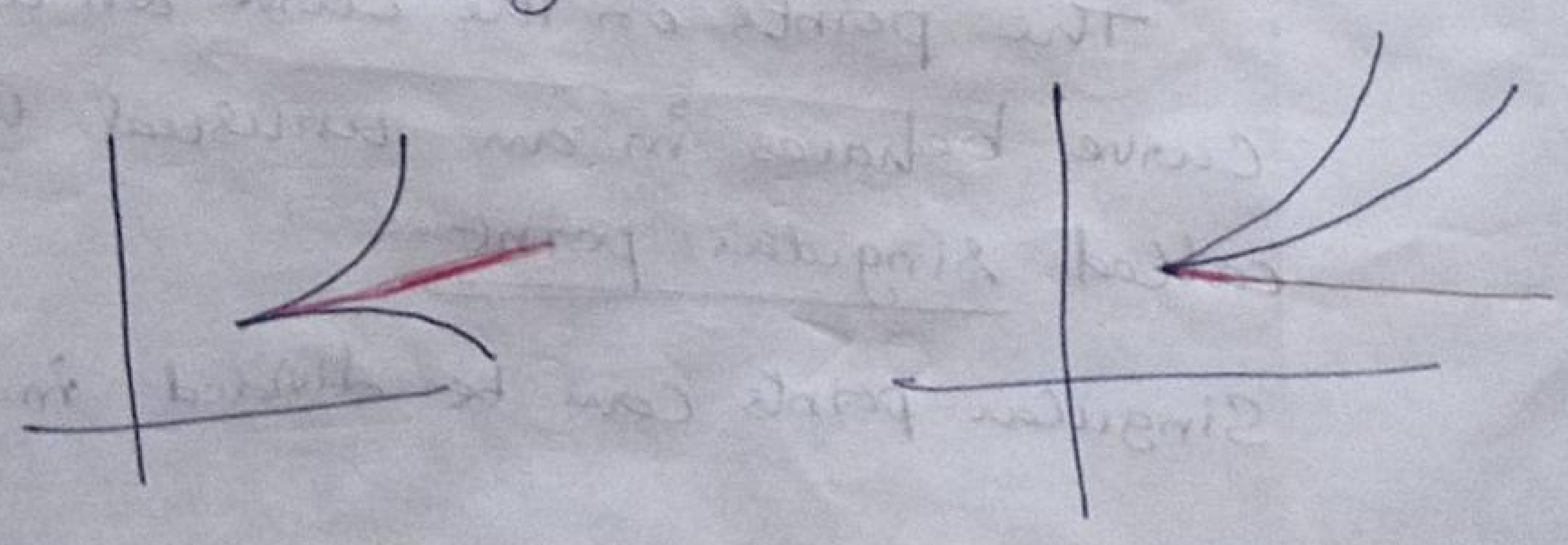
Node

A double point is called a node if the two branches of the curve passing through this point are real and the tangents to them are distinct



Cusp

A double point is called a cusp if the two branches of the curve passing through this point are real and tangents to them are coincident



A cusp can be single or double.

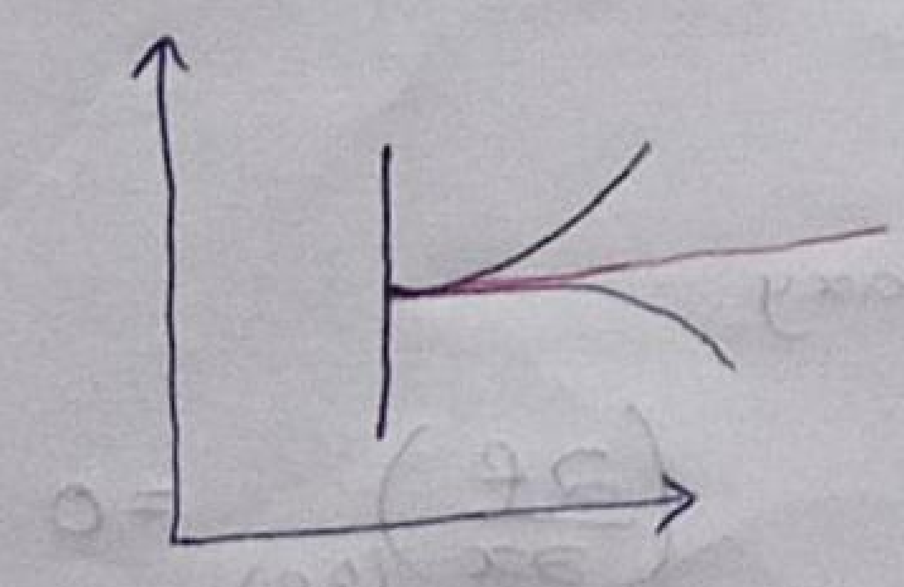
If the curve lies entirely on one side of the normal, it is a single cusp.

If it lies on both sides of the normal it is a double cusp.

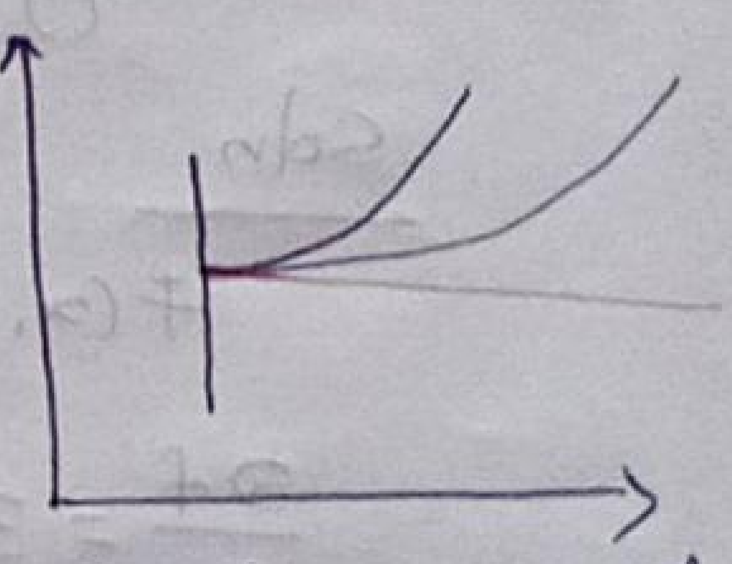
A cusp can be of first or second species.

If the two branches of the curve lie on opposite side of the common tangent the cusp is of ~~the~~ first species.

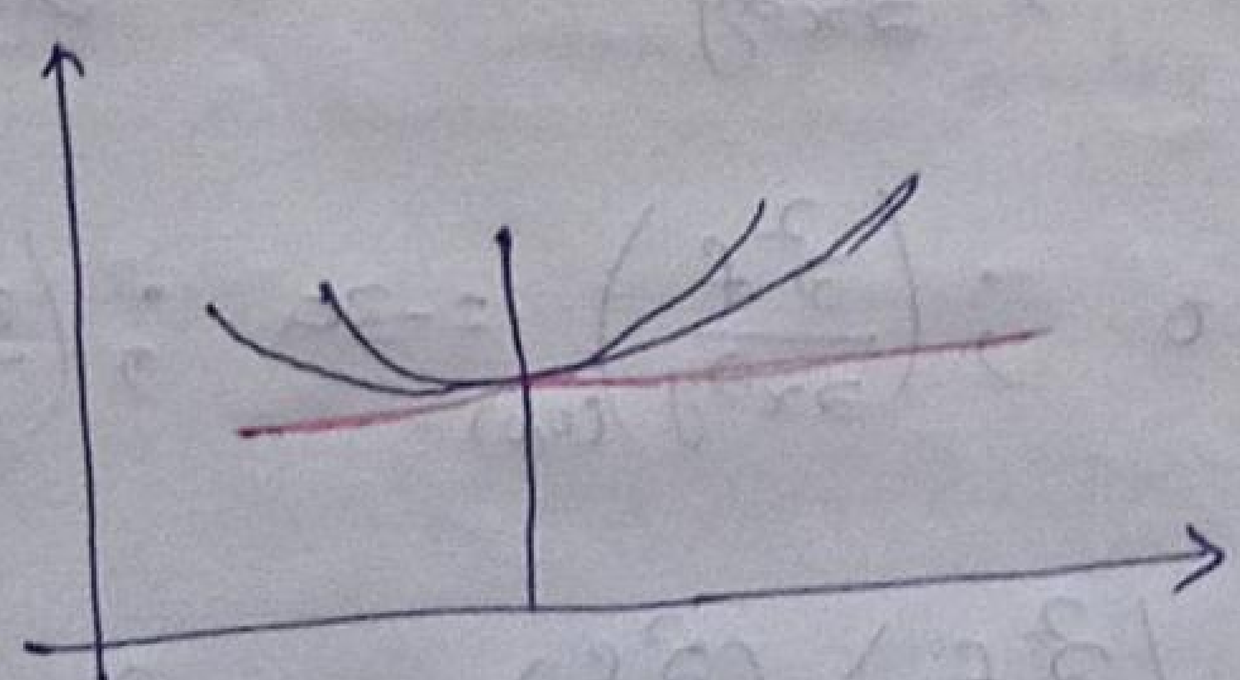
If the branches lies on the same side of the common tangent, the cusp is of second species



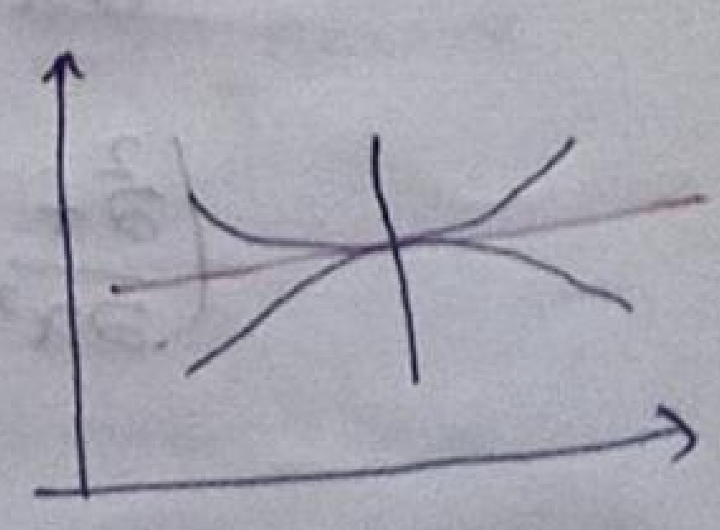
Single cusp of first species



Single cusp of second species.



Double cusp of second species



Double cusp of first species.

Conditions for the existence of Double points

The curve $f(x,y)=0$ has double point if

$\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

The double point will be a node, cusp or conjugate point according as

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) >0, =0, <0$$

ie according as the tangents at the double point are different, coincident or ~~imaginary~~ imaginary.

---

① find the nature of double point of the following curve at origin $x^3+y^3-3axy=0$

Soln

$$f(x,y) = x^3+y^3-3axy$$

$$\frac{\partial f}{\partial x} = 3x^2-3ay \qquad \left(\frac{\partial f}{\partial x}\right)_{(0,0)}=0$$

$$\frac{\partial f}{\partial y} = 3y^2-3ax \qquad \left(\frac{\partial f}{\partial y}\right)_{(0,0)}=0$$

$$\frac{\partial^2 f}{\partial x^2}=6x \; ; \; \frac{\partial^2 f}{\partial x \partial y}=-3a \; ; \; \frac{\partial^2 f}{\partial y^2}=6y$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)}=0 \; ; \; \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)}=-3a \; ; \; \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)}=0$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{(0,0)} - \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} \cdot \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 9a^2$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} - \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} \cdot \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} >0$$

∴ The curve has a node at origin.

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(2) Examine the position and nature of double points of the following curve

$$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$$

soln

The equation of the curve can be written as

$$a^2y^2 - b^2x^2 = x^2y^2$$

$$f(x,y) = a^2y^2 - b^2x^2 - x^2y^2$$

$$\frac{\partial f}{\partial x} = -2b^2x - 2xy^2 = -2x(b^2+y^2)$$

$$\frac{\partial f}{\partial y} = 2a^2y - 2x^2y = 2y(a^2-x^2)$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow y = 0 \ ; \ x = \pm a.$$

$\therefore$ The possible double points are $(0,0)$ and $(a,0)$, $(-a,0)$

But $(a,0)$, $(-a,0)$ does not satisfy the equation of the curve

$\therefore$ The only double points is $(0,0)$

$$\frac{\partial^2 f}{\partial x^2} = -2(b^2+y^2) \ ; \ \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = -2b^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4xy \ ; \ \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 2(a^2-x^2) \ ; \ \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 2a^2$$

$$\therefore \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{(0,0)} - \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} \cdot \left(\frac{\partial^2 f}{\partial y^2}\right) = 4a^2b^2 > 0$$

$\therefore$ The curve has a node ~~⇒~~ at $(0,0)$

③ Examine the position and nature of double points of the following curve $y^3 = (x-a)^2(2x-a)$

Soln

$f(x,y) = y^3 - (x-a)^2(2x-a)$

$\frac{\partial f}{\partial x} = -[(x-a)^2 \cdot 2 + 2(x-a)(2x-a)]$

$\frac{\partial f}{\partial x} = -2(x-a)(3x-2a)$

$\frac{\partial f}{\partial y} = 3y^2$

$\frac{\partial f}{\partial x} = 0 \Rightarrow x = a, \quad x = \frac{2a}{3}$

$\frac{\partial f}{\partial y} = 0 \Rightarrow y = 0$

$\therefore$ The possible double points are $(a,0)$ & $(\frac{2a}{3}, 0)$

But $(\frac{2a}{3}, 0)$ does not satisfy the eqn of the curve

$\therefore$ The only double point is $(a,0)$

$\frac{\partial^2 f}{\partial x^2} = -2[(x-a) \cdot 3 + (3x-2a)]$; $\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,0)} = -2a$

$\frac{\partial^2 f}{\partial x \partial y} = 0$; $\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,0)} = 0$

$\frac{\partial^2 f}{\partial y^2} = 6y$; $\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,0)} = 0$

$\therefore \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{(a,0)} - \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,0)} \cdot \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,0)} = 0$

$\therefore$ The curve has a cusp at $(a,0)$

Shifting the origin to $(a,0)$, the eqn of the curve becomes [replace $x = x+a$ & $y = y+0$]

$y^3 = x^2(2x+a)$ —— ①

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For find the tangent at origin, equating the lowest degree term to zero

Tangents at origin are given by $x^2=0$

Also from eqn ① $y=\left[x^2(2x+a)\right]^{1/3}$

when x is very small (ie near origin) higher power of x ie x^3 can be neglected

Then the eqn ① becomes

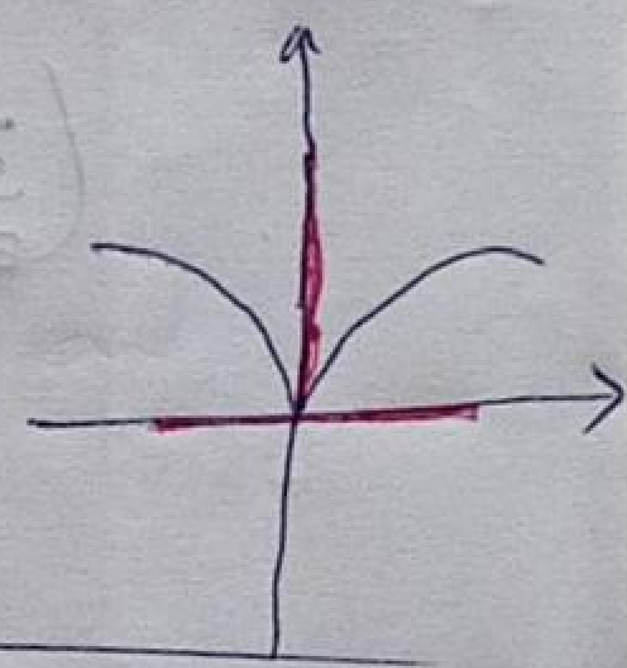
$$y^3 = ax^2 \Rightarrow x = \pm\sqrt{\frac{y^3}{a}}$$

when $y>0$ there are two values of x. One positive and the other negative.

$\therefore$ The cusp is of first species.

Also when $y<0$, x is imaginary

$\therefore$ The cusp is single.



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④ ② Find the position and nature of the double points ~~on~~ on the curve

$$x^2y^2=(a+y)^2(b^2-y^2)$$

And distinguish between the cases when $a>, =, < b$.

Soln

$$f(x,y)=x^2y^2-(a+y)^2(b^2-y^2)$$

$$\frac{\partial f}{\partial x}=2xy^2$$

$$\frac{\partial f}{\partial y}=2x^2y-\cancel{2(}\left[(a+y)^2(-2y)+(b^2-y^2)\cdot 2(a+y)\right]$$

$$=2x^2y-2(a+y)\left[-y(a+y)+(b^2-y^2)\right]$$

$$=2x^2y-2(a+y)\left[-2y^2-ay^2+b^2\right]$$

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$$\frac{\partial f}{\partial x} = 0 \Rightarrow \quad x=0 \quad (\text{or}) \quad y=0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \quad y = -a \quad [\because x=0]$$

$\therefore$ Double point is $(0, -a)$

$$\frac{\partial^2 f}{\partial x^2} = 2y^2 \quad ; \quad \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,-a)} = 2a^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4xy \quad ; \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,-a)} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^2 - 2\left[(a+y)(-4y-a) + (-2y^2 - ay + b^2)\right]$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,-a)} = -2(b^2 - a^2)$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,-a)} - \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,-a)} \cdot \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,-a)} = 0 - 2a^2(-2)(b^2 - a^2)$$

$$= 4a^2(b+a)(b-a)$$

$4a^2(b+a)(b-a) > 0$ if $b > a$.

$4a^2(b+a)(b-a) = 0$ if $b = a$

$4a^2(b+a)(b-a) < 0$ if $b < a$.

$\therefore$ The double point is node, cusp or conjugate point according as $b > a$, $b = a$, $b < a$.

Intersection of a curve with its asymptotes

(1) Show that the asymptotes of the cubic

$x^2y - xy^2 + xy + y^2 + x - y = 0$

cut the curve again in three points which lie on the line

$x + y = 0$

Soln

The equation of the curve can be written as

$$(x^2y - xy^2) + (xy + y^2) + (x - y) = 0 \quad \text{--- (1)}$$

$\varphi_3(x) = x^2y - xy^2$

$\varphi_2(x) = xy + y^2$

Put $x = 1 \quad y = m$

$\varphi_3(m) = m - m^2$

$\varphi_2(m) = m + m^2$

$\varphi_3'(m) = 1 - 2m$

For finding m we put $\varphi_3(m) = 0$

$$m - m^2 = 0$$

$$m(1 - m) = 0$$

$$\boxed{m = 0 \quad m = 1}$$

For finding c

$$c = -\frac{\varphi_2(m)}{\varphi_3'(m)}$$

$$= -\left(\frac{m + m^2}{1 - 2m}\right)$$

when $m = 0$

$$\boxed{c = 0}$$

The corresponding asymptotes is ~~y=x~~ $\boxed{y = 0}$

when $m=1$

$$c = -\left(\frac{1+(1)^2}{1-2}\right)$$

$$\boxed{c = 2}$$

$\therefore$ The corresponding asymptotes is $y = x+2$

The asymptotes parallal to the axes are got by equating the coefficients of x^2 and y^2 to zero

$\boxed{y=0}$ and $(1-x)=0$

$$\boxed{x=1}$$

$\therefore$ The asymptotes are

$$\boxed{y=0\ ;\ x-y+2=0\ ;\ x-1=0}$$

so the combined eqn of

$$y(y-x-2)(x-1)=0$$

$$x^2y - xy^2 + xy + y^2 - 2y = 0 \quad \text{—— ②}$$

① - ②

$$x^2y - xy + xy + y^2 + x - y - x^2y + xy^2 - xy - y^2 + 2y = 0$$

$$\boxed{x+y = 0}$$

proved

Also

Asymptotes cuts the given curve is $n(n-2)$

ie $3(3-2) = 3$

② Show that the asymptotes of the quadratic
$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0$
cut the curve in the eight points which lie on a circle.

Soln

Given eqn is

$$x^4-13x^2y^2+36y^4+5x^2y-5xy^2-30y^3+xy+7y^2-1=0 \quad -①$$

put $x=1,\ y=m$ in highest degree terms

$\varphi_4(m)=1-13m^2+36m^4=0$

$36m^4-9m^2-4m^2+1=0$

$9m^2(4m^2-1)-1(4m^2-1)=0$

$(4m^2-1)(9m^2-1)=0$

$4m^2-1=0$	$9m^2-1=0$
$m^2=\frac{1}{4}$	$m^2=\frac{1}{9}$
$\boxed{m=-\frac{1}{2},\frac{1}{2}}$	~~$m=-\frac{1}{9},\frac{1}{9}$~~ $\quad \boxed{m=-\frac{1}{3},\frac{1}{3}}$

$\varphi_3(m)=5m-5m^2-30m^3$

$\varphi_4'(m)=-26m+144m^3$

$$c=-\frac{\varphi_3(m)}{\varphi_4'(m)}=-\left(\frac{5m-5m^2-30m^3}{-26m+144m^3}\right)$$

when $m=\frac{1}{2}$

$$c=-\left(\frac{5/2-\frac{5}{4}-\frac{30}{8}}{-26(\frac{1}{2})+144(\frac{1}{8})}\right)$$

$$=-\left(\frac{5/2-5/4-30/8}{-13+18}\right)$$

$$=-\left(\frac{10-5-15}{4\times5}\right)$$

$\boxed{c=1/2}$

$\boxed{m=\frac{1}{2}}$	$\boxed{c=\frac{1}{2}}$
$m=-\frac{1}{2}$	$c=0$
$m=1/3$	$c=0$
$m=-\frac{1}{3}$	$c=1/3$

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$\therefore$ The asymptotes are

$x+2y=0$

$x-2y+1=0$

$x-3y=0$

$x+3y-1=0$

$\therefore$ The joint eqn of these asymptotes

$(x+2y)(x-2y+1)(x-3y)(x+3y-1)=0$

$x^4-13x^2y^2+36y^4+5x^2y-5xy^2-30y^3-x^2+xy+6y^2=0$ —②

① - ②

$x^4-13x^2y^2+36y^4+5x^2y-5xy^2-30y^3+xy+7y^2-1$

$-x^4+13x^2y^2-36y^4-5x^2y+5xy^2+30y^3+x^2-xy-6y^2=0$

$-x^2+6y^2-7y^2+1=0$

$-x^2-y^2=-1$

$x^2+y^2=1$

$\boxed{x^2+y^2=1^2}$

This is eqn of circle radius is 1

No. of points $=n(n-2)$

$=4(4-2)$

$=8$

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③ show that the four asymptotes to the curve

$(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0$

cut the curve again in eight points which lies on the circle $x^2+y^2=1$

Soln

Given

$(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0$

put $x=1$ $y=m$ in highest degree terms

$\phi_4(m)=(1-m^2)(m^2-4)=0$

$\phi_3(m) = 6-5m-3m^2+2m^2$

$\phi_2(m) = -1+3m$

$\phi_4'(m)=-2m(m^2-4)+(1-m^2)(2m)$

$\phi_4'(m)=10m-4m^3$

Let $\phi_4(m)=0$

$(1-m^2)(m^2-4)=0$

$m=1,-1,2,-2$

$c=-\dfrac{\phi_3(m)}{\phi_4'(m)}$

$=-\left(\dfrac{6-5m-3m^2+2m^3}{10m-4m^3}\right)$

For $m_1=1$

$c=-\left(\dfrac{6-5-3+2}{10-4}\right)$

$\boxed{c=0}$

$y=x$

$\boxed{y-x=0}$

For $m_2=-1$

$c=-\left(\dfrac{6+5-3-2}{-10+4}\right)$

$\boxed{c=1}$

$y=-x+1$

$\boxed{y+x-1=0}$

For $m_3 = 2$

$$c = -\left(\frac{6-10+2+16}{20-32}\right)$$

$c = 0$

$\boxed{y = 2x}$

$\boxed{y - 2x = 0}$

For $m_4 = -2$

$$c = \frac{6+10-12-16}{-20+32}$$

$c = 1$

$y = -2x+1$

$\boxed{y + 2x - 1 = 0}$

$$(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0 \quad \text{(1)}$$

$$(y-x)(y+x-1)(y-2x)(y+2x-1)=0 \quad \text{——— (2)}$$

(1) - (2) $\Rightarrow x^2+y^2=1$

which is a circle of unit ~~cir~~ radius and cut the ~~eight points~~ curve at $n(n-2)$ points

ie $4(4-2)$ points

$= 4(2)$

$= 8$ points.

Tracing of curves

To trace the graph of a curve whose equation is given in cartesian coordinates it is better to adopt a method as detailed below.

The first consideration is the symmetry of a curve with respect to certain lines or points. The various kinds of symmetry arising from the form of the equation are as follows.

① If the equation of the curve contains only even powers of y, the curve is symmetrical with respect to the x-axis. for if (x,y) be a point in the curve $(x,-y)$ will also be on the curve.

Example:

$y^2 = 4ax$, $xy^2 = 4a^2(2a-x)$; $y^2(2a-x) = x^3$

② If the eqn contain only even power of x the curve is symmetrical about the y-axis.

Example

$x^4 = a^2(x^2-y)$; $x^2 = y^2\left(\frac{y+a}{y-a}\right)$

If the eqn contains only even powers of x and even power of y, the curve is symmetrical with respect to both axes

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Example

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $(x^2-a^2)(y^2-b^2)=a^2b^2$;

$x^2y^2=a^2(x^2-y^2)$

(b) If the equation be not altered when $-x$ and $-y$ are written for x and y, the curve is symmetrical with respect to the origin in opposite quadrants

Example

$xy=c^2$; $x^4=a^2(x^2-y^2)$; $x^3+y^3=a^2x$.

(c) If the equation be not altered when x and y are interchanged, the curve is symmetrical with respect to a line bisecting XOY ie $y=x$.

Example

$xy=c^2$; $x^3+y^3=3axy$

(2) Special points on the curve

(a) Find the points where the curve crosses the x-axis. To find these points, Substitute 0 for y in the equation and solve the resulting equation in x. IIIly find the points where the curve crosses the y-axis

(b) If the equation of the curve does not involve the constant term, it passes through the origin

(c) Find the values of x which will make the values of y imaginary. Let those values be $x=\alpha$ and $x=\beta$. Then no real part of the curve will lie between the lines $x=\alpha$ and $x=\beta$

III^ly find the values of y which will make the values of x imaginary. If those value of y are γ and δ, the curve will not lie between the lines $y=\gamma$ and $y=\delta$

(d) Find the values of x for which the value of y is infinite and the value of y for which the value of x is infinite. From these value we can find the equations to the asymptotes of the curve parallel to the coordinate axes.

(e) If the equation of the curve is given as in implicit equation it is better to express the curve in the form $y=f(x)$.

(3)

(a) From the equation of the curve find the value of $\frac{dy}{dx}$. The value of $\frac{dy}{dx}$ at a point gives the slope of the curve at that point.

(b) find $\frac{d^2y}{dx^2}$ from this we can determine the range within which the curve is concave upwards or concave downwards and the points of inflexion

(c) find the points where the curve attains its maximum (or) minimum if any.

Problems

(1) Trace the curve whose equation is $y = \frac{x^2+1}{x^2-1}$

(1) Symmetry

Since the terms involving x are even powers the curve is symmetrical about y-axis.

(2) Special points

(a) $y=-1$ when $x=0$; when $y=0$, x has imaginary value

∴ The curve will not intersect the x-axis but will cross the y-axis at the point $(0,-1)$

(b)

We can write the equation of the curve as

$$x^2 = \frac{y+1}{y-1}$$

when the value of y lies between $+1$ and -1 the value of x^2 is negative

∴ For those values of y, x is imaginary

∴ The curve does not lie b/w the lines $y=1$ and $y=-1$

(c) y tends to infinity when x tends to ~~±~~ $+1$ (or) to -1

∴ $x=1$ and $x=-1$ are asymptotes of the curve.

x tends to infinite when y tends to 1

∴ $y=1$ is an asymptotes

(3) $y = \dfrac{x^2+1}{x^2-1}$

$$\frac{dy}{dx} = -\frac{4x}{(x^2-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{4(x^2+1)}{(x^2-1)^3}$$

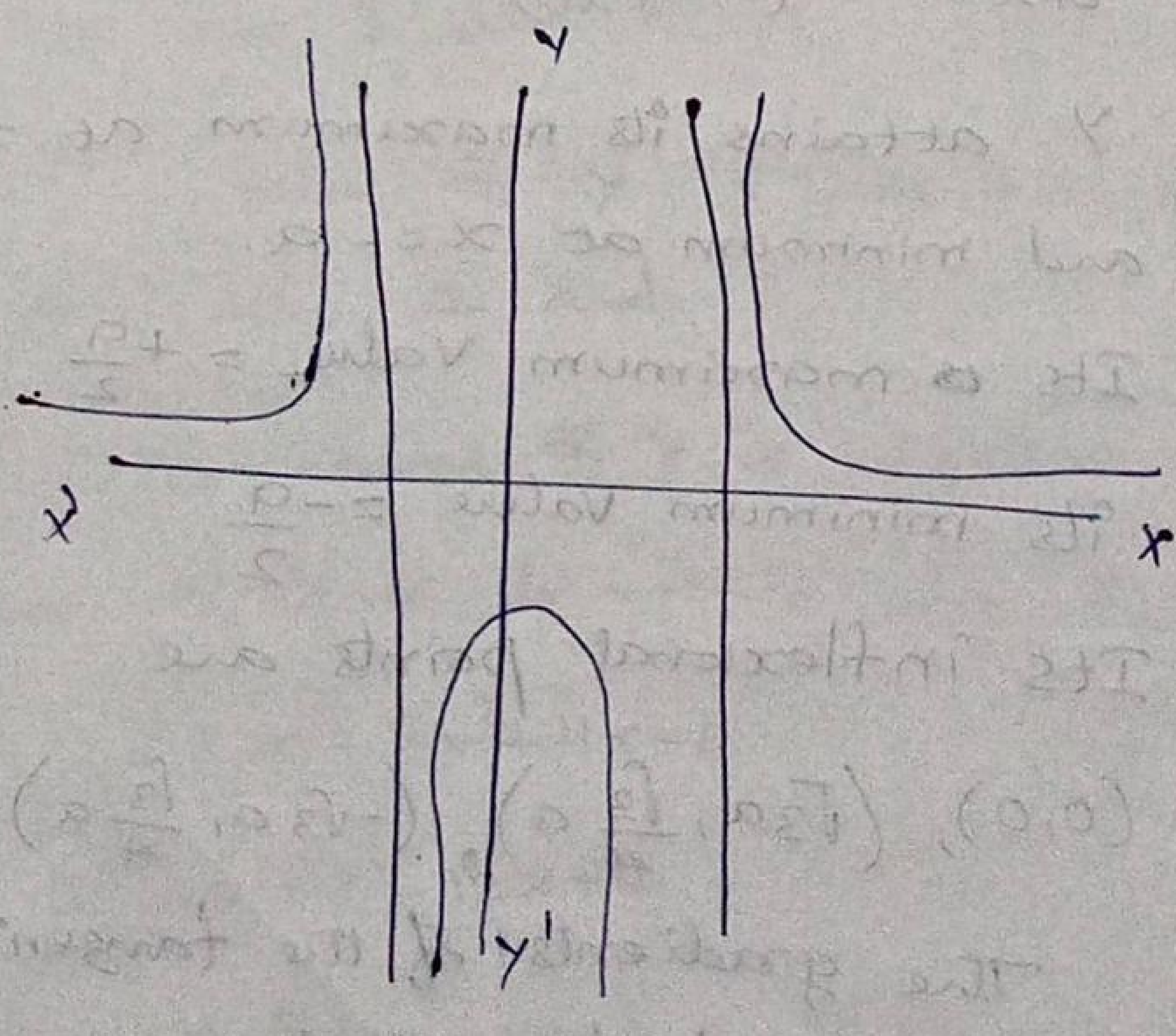
$\dfrac{dy}{dx} = 0$ when $x = 0$

$\therefore$ The curve attains its maximum when $x=0$ and the maximum value is -1.

(4) Giving different values for x, calculate the values for y

x:	1	2	3	4	5	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
y:	∞	$\frac{3}{5}$	$\frac{5}{4}$	$\frac{17}{16}$	$\frac{13}{12}$	$-\frac{17}{15}$	$\frac{-5}{3}$	$\frac{-25}{7}$.

The shape of the curve known from these is as below



(2) Trace of curve $(a^2+x^2)y = a^2x$.

Solution

Symmetry

The equation of the curve is unaltered after substituting $-x$ and $-y$ for x and y. Hence the curve is symmetrical in the opposite quadrants

Special points

The curve passes through the origin.

$$y = \frac{a^2x}{a^2+x^2}$$

As x increases, the values of y decreases and when x tends to $\pm\infty$, y tends to 0.

$\therefore$ $y=0$ is an asymptotes, to the curve.

$$\frac{dy}{dx} = \frac{a^2(a^2-x^2)}{(a^2+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2a^2x(3a^2-x^2)}{(a^2+x^2)^3}$$

y attains its maximum at $x=a$ and minimum at $x=-a$.

Its maximum value $= +\frac{a}{2}$.

its minimum value $= -\frac{a}{2}$

Its inflexional points are

$(0,0)$, $\left(\sqrt{3}a, \frac{\sqrt{3}}{4}a\right)$, $\left(-\sqrt{3}a, \frac{\sqrt{3}}{4}a\right)$

The gradients of the tangents at these points are 1, $-\frac{1}{8}$ and $-\frac{1}{8}$

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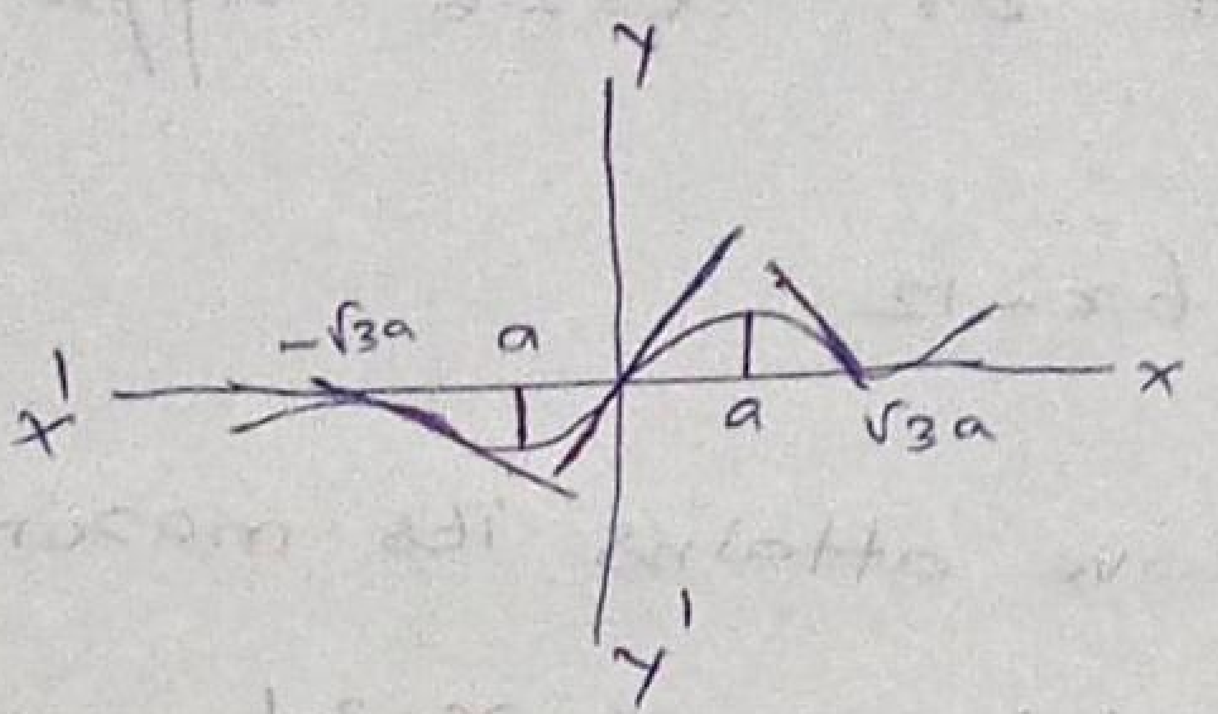
The slope of the tangents at the origin $= 45^\circ$



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(3) Trace the curve $y=(x-1)(x-2)(x-3)$

Soln

There is no symmetry in the curve.

Special points

The curve crosses the x-axis at 1,2 and 3 and crosses the y-axis at -6

As the value of x increases, the value of y increases for $x>3$

when x tends to infinity, y also tends to ∞

The value of y is ~~positi~~ positive when x lies b/w 1 and 2.

The value of y is negative when x lies b/w 2 and 3

The value of y is negative when $x<1$

when x tends to $-\infty$, y also tends to $-\infty$

$$y=x^3-6x^2+11x-6$$

$$\frac{dy}{dx}=3x^2-12x+11$$

when $3x^2-12x+11=0$

$$x=\frac{12\pm\sqrt{144-132}}{6}$$

$\therefore x = 1.4$ or $x = 2.6$ approx.

$$\frac{d^2y}{dx^2} = 6x - 12$$

$\therefore$ The curve attains its maximum at $x=1.4$ and its minimum at $x=2.6$

Its maximum value $= 0.384$

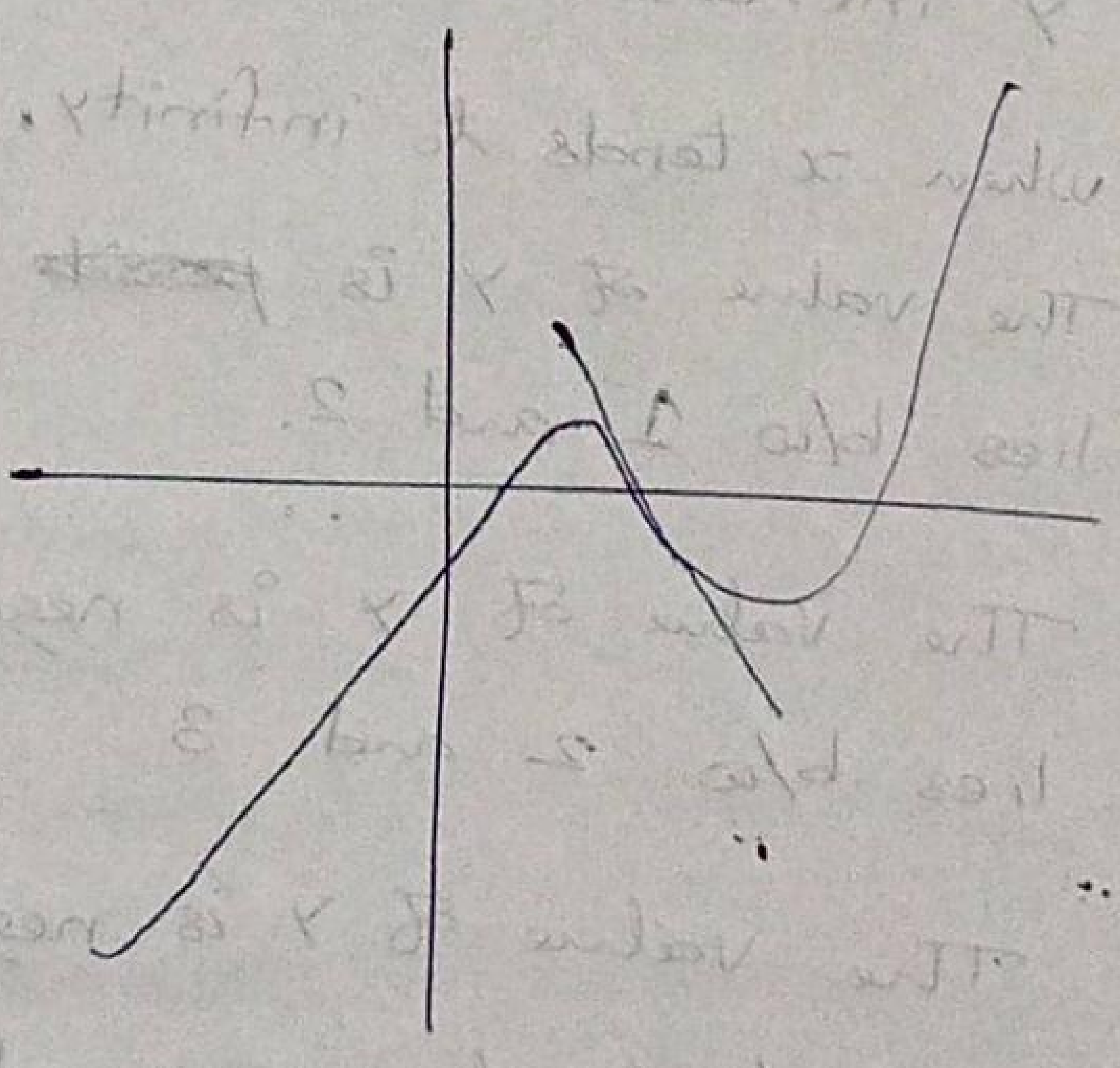
Its minimum value $= -0.384$

$\therefore$ The gradients of the tangent at the points $x=1$, $x=2$, $x=3$ are respectively $2, -1, +1$.

when $x=2$ $\frac{d^2y}{dx^2}=0$

The curve has inflexional points at $(2,0)$

The shape of the curve is given below



Unit-V

Reduction formula

1. Reduction formula for $\int x^n e^{ax} dx$ where n is a positive integer.

<u>Soln</u>

Let $I_n = \int x^n e^{ax} dx$ —— ①

$u = x^n$

$\frac{du}{dx} = n x^{n-1}$

$du = n x^{n-1} dx$

$dv = e^{ax} dx$

$\int dv = \int e^{ax} dx$

$v = \frac{e^{ax}}{a}$

Using integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$I_n = \int x^n e^{ax} dx$$

$$= x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot n x^{n-1} dx$$

$$= x^n \frac{e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$\boxed{I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a} I_{n-1}}$$ by using ①

which is the required formula.

Put $n=0$ in ①

$$I_0 = \int x^0 e^{ax} dx = \int e^{ax} dx = \frac{e^{ax}}{a}$$

$$I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a}\left[x^{n-1} \frac{e^{ax}}{a} - \frac{n-1}{a} I_{n-2}\right]$$

(4) Reduction formula for $\int x^n \cos ax\,dx$ n is a +ve integer

Soln

let $I_n = \int x^n \cos ax\,dx$

$u = x^n$

$du = n x^{n-1} dx$

$dv = \cos ax\,dx$

$v = \frac{\sin ax}{a}$

$\int u\,dv = uv - \int v\,du$

$\int \cos ax\,dx = \frac{\sin ax}{a}$

$\int \sin ax\,dx = -\frac{\cos ax}{a}$

$I_n = \int x^n \cos ax\,dx$

$= x^n \frac{\sin ax}{a} - \int \frac{\sin ax}{a} n x^{n-1} dx.$

$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} \sin ax\,dx.$

$= \frac{x^n \sin ax}{a} - \frac{n}{a}\left\{ x^{n-1}\left(-\frac{\cos ax}{a}\right) - \int -\frac{\cos ax}{a}(n-1)x^{n-2}dx\right\}$

$= \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1}\cos ax - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax\,dx$

$$\boxed{I_n = \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1}\cos ax - \frac{n(n-1)}{a^2} I_{n-2}}$$

which is the required reduction formula

If n is even

$I_0 = \int x^0 \cos ax\,dx$

$$\boxed{I_0 = \frac{\sin ax}{a}}$$

If n is odd

$I_1 = \int x \cos ax\,dx$

$= x\left(\frac{\sin ax}{a}\right) - \int \frac{\sin ax}{a}dx$

$$\boxed{I_1 = \frac{x \sin ax}{a} + \frac{\cos ax}{a^2}}$$

H.W

(5) find reduction formula for $\int x^n \sin ax\,dx$ where n is a +ve integer.

4 Reduction formula for $\int x^m (\log x)^n dx$.

let $I_{m,n} = \int x^m (\log x)^n dx$

$u = (\log x)^n$

$du = n \dfrac{(\log x)^{n-1}}{x} dx$

$dv = x^m dx$.

$v = \dfrac{x^{m+1}}{m+1}$

$$I_{m,n} = \left[(\log x)^n \frac{x^{m+1}}{m+1} \right] - \int \frac{x^{m+1}}{m+1} \cdot n \frac{(\log x)^{n-1}}{x} dx.$$

$$= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx.$$

$$\boxed{I_{m,n} = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} I_{m,n-1}}$$

5 Reduction formula for $\int \sin^n x \, dx$.

Soln

$$I_n = \int \sin^n x \, dx$$

$$= \int \sin^{n-1} x \sin x \, dx.$$

$u = \sin^{n-1} x$

$du = (n-1) \sin^{n-2} x \cos x \, dx$

$dv = \sin x$

$v = -\cos x$.

$$I_n = \sin^{n-1} x (-\cos x) - \int (n-1) \sin^{n-2} x \cos x (-\cos x) dx.$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx.$$

$$= -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore\ 1+(n-1)I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2}$$

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⑥ Reduction formula for $\int \cos^n x\,dx$.

<u>Soln</u>

$$I_n = \int \cos^n x\,dx$$

$$= \int \cos^{n-1}x\cos x\,dx$$

$$= \cos^{n-1}x\sin x - \int (n-1)\cos^{n-2}x(-\sin x)\sin x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}I_{n-2}$$

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⑦ Reduction formula for $\int_0^{\pi/2} \sin^n x\,dx$.

<u>Solution</u>

$$I_n = \int_0^{\pi/2} \sin^n x\,dx.$$

$$= \int_0^{\pi/2} \sin^{n-1}x\sin x\,dx.$$

$u = \sin^{n-1}x$	$v = \sin x\,dx$
$du = (n-1)\sin^{n-2}x\cos x\,dx$	$dv = -\cos x.$

$$I_n = \left[-\cos x\sin^{n-1}x\right]_0^{\pi/2} + \int_0^{\pi/2}\cos x(n-1)\sin^{n-2}x\cos x\,dx.$$

(8) Reduction formula for $\int_0^{\pi/2} \cos^n x\,dx$.

Soln

$$I_n = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \int_0^{\pi/2} \cos^{n-1} x \cos x\,dx$$

Let $u = \cos^{n-1} x$ | $dv = \cos x$

$du = (n-1)\cos^{n-2} x(-\sin x\,dx$ | $v = \sin x$

$$I_n = \left[\cos^{n-1} x \sin x\right]_0^{\pi/2} + \int_0^{\pi/2} \sin x (n-1) \cos^{n-2} x \sin x\,dx.$$

$$= 0 + (n-1)\int_0^{\pi/2} \cos^{n-2} x(1-\cos^2 x)dx$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$(1+n-1) I_n = (n-1) I_{n-2}$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

to Proceeding as in the last example

when n is even

$$\boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}}$$

when n is odd

$$\boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}}$$

$$= 0 + (n-1)\int_0^{\pi/2} \sin^{n-2}x \cos^2 x\,dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2}x (1-\sin^2 x)\,dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2}x\,dx - (n-1)\int_0^{\pi/2} \sin^n x\,dx$$

$$\therefore I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$(1+n-1) I_{n-2} = (n-1) I_{n-2}$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

Evaluation of I_n

Case 1 : n is even

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot I_{n-4}$$

Proceeding in this ways

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} I_0$$

But $I_0 = \int_0^{\pi/2} dx = \pi/2$

$$\boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}}$$

Case 2 : n is odd

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} I_1$$

But $I_1 = \int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = 1$

$$\boxed{\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}}$$

Case 1 Suppose m is even

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}$$

$$I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

$$\vdots$$

$$I_{4,n} = \frac{3}{n+4} I_{2,n}$$

$$I_{2,n} = \frac{1}{n+2} I_{0,n}$$

Multiplying all these result

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{3}{n+4} \cdot \frac{1}{n+2} I_{0,n}$$

$$I_{0,n} = \int_0^{\pi/2} \cos^n x \, dx$$

$$= \frac{n-1}{m} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{if } n \text{ is even.}$$

$$= \frac{n-1}{m} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \quad \text{if } n \text{ is odd}$$

$\therefore$ If m is even and n is even

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{3}{n+4} \cdot \frac{1}{n+2} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-1} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{—(1)}$$

$\therefore$ If m is even and n is odd

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{3}{n+4} \cdot \frac{1}{n+2} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \quad \text{—(2)}$$

$\boxed{\text{Case 2}}$

Suppose m is odd

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

$$I_{m-2,n} = \frac{m-1}{m+n-2} I_{m-4,n}$$

$$\vdots$$

$$I_{5,n} = \frac{4}{n+5} I_{3,n}$$

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(9) Reduction formula for $\int \tan^n x \, dx$.

Soln

$$I_n = \int \tan^n x \, dx$$

$$= \int \tan^{n-2} x \, dx \, (\sec^2 x - 1) dx.$$

$$= \int \tan^{n-2} x \; d(\tan x) - \int \tan^{n-2} x \, dx$$

$$\boxed{I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}}$$

(10) Reduction formula for $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$.

$$I_{m,n} = \int_0^{\pi/2} \sin^{m-1} x \sin x \cos^n x \, dx.$$

$$= \int_0^{\pi/2} -\sin^{m-1} x \cos^n x \; d(\cos x)$$

$$= \left[-\sin^{m-1} x \frac{\cos^{n+1} x}{n+1}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos^{n+1} x}{n+1} (m-1) \sin^{m-2} x \cos x \, dx$$

$$= \frac{m-1}{n+1} \int_0^{\pi/2} \cos^{n+2} x \sin^{m-2} x \, dx$$

$$= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x (1 - \sin^2 x) dx$$

$$= \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \cos^n x \, dx - \frac{m-1}{n+1} \int_0^{\pi/2} \sin^m x \cos^n x \, dx.$$

$$I_{m,n} = \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$\left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$$

$$\left(\frac{m+n}{n+1}\right) I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$$

$$\boxed{I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}}$$

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$$I_{3,n} = \frac{2}{n+3} I_{1,n}$$

Multiplying all these results

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{4}{n+5} \cdot \frac{2}{n+3} I_{1,n}$$

$$I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x \, dx.$$

$$= \int_0^{\pi/2} -\cos^n x \, d(\cos x) = \left[-\frac{\cos^{n+1} x}{n+1} \right]_0^{\pi/2} = \frac{1}{n+1}$$

$$\boxed{I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{4}{n+5} \cdot \frac{2}{n+3} \cdot \frac{1}{n+1}}$$

H.W $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ P.T. $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$ and hence evaluate $I_{m,n}$

(11) If $I_n = \int_0^{\pi/2} x^n \cos x \, dx$. S.T. $I_n + n(n-1) I_{n-2} = \left(\frac{\pi}{2}\right)^n$ for $n > 1$

soln.

$$I_n = \int_0^{\pi/2} x^n \cos x \, dx.$$

$$= \left(x^n \sin x\right)_0^{\pi/2} - \int_0^{\pi/2} n x^{n-1} \sin x \, dx.$$

$$= \left(\frac{\pi}{2}\right)^n - n \int_0^{\pi/2} x^{n-1} \sin x \, dx.$$

$$= \left(\frac{\pi}{2}\right)^n - n \left[\left(-x^{n-1} \cos x\right)_0^{\pi/2} + \int_0^{\pi/2} (n-1) x^{n-2} \cos x \, dx \right]$$

$$= \left(\frac{\pi}{2}\right)^n + 0 - n(n-1) I_{n-2}$$

$$\therefore I_n + n(n-1) I_{n-2} = \left(\frac{\pi}{2}\right)^n$$

(12) Reduction formula for $\int_0^{\pi/2} x^n \sin x\, dx$.

Soln

Let $I_n = \int_0^{\pi/2} x^n \sin x\, dx$

$$= \left[-x^n \cos x\right]_0^{\pi/2} + \int_0^{\pi/2} n x^{n-1} \cos x\, dx$$

$$= 0 + n \int_0^{\pi/2} x^{n-1} \cos x\, dx.$$

$$= n\left[x^{n-1} \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} (n-1)\, x^{n-2} \sin x\, dx$$

$$= n\left[\left(\tfrac{\pi}{2}\right)^{n-1} - (n-1) I_{n-2}\right]$$

$$\boxed{I_n + n(n-1) I_{n-1} = n\left(\frac{\pi}{2}\right)^{n-1}}$$

(13) Reduction formula for $\int_0^{\infty} e^{-x} x^n\, dx$.

Soln

Let $I_n = \int_0^{\infty} e^{-x} x^n\, dx$

$$= \left[-e^{-x} x^n\right]_0^{\infty} + \int_0^{\infty} n x^{n-1} e^{-x}\, dx.$$

$$= 0 + n \int_0^{\infty} x^{n-1} e^{-x}\, dx.$$

$$I_n = n\, I_{n-1}$$

Also

$$I_{n-1} = (n-1) I_{n-2}$$

$$I_{n-2} = (n-2) I_{n-3}$$

$$I_1 = 1 \cdot I_0$$

Multiplying all these results

$$I_n = n(n-1)(n-2) \cdots 2 \cdot 1 \cdot I_0$$

$$I_0 = \int_0^{\infty} e^{-x}\, dx = \left[e^{-x}\right]_0^{\infty} = 1$$

$$I_1 = n!$$

Unit- V

Reduction formula.

1. $\int x^n e^{ax}\,dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} I_{n-1}$

2. $\int x^n \cos ax\,dx = \dfrac{x^n \sin ax}{a} + \dfrac{n}{a^2} x^{n-1}\cos ax - \dfrac{n(n-1)}{a^2} I_{n-2}$

3. $\int x^n \sin ax\,dx = \dfrac{x^n \cos ax}{a} + \dfrac{n}{a^2} x^{n-1}\sin ax - \dfrac{n(n-1)}{a^2} I_{n-2}$

4. $\int \sin^n x\,dx = -\dfrac{\sin^{n-1}x\cos x}{n} + \dfrac{(n-1)}{n} I_{n-2}$

5. $\int \cos^n x\,dx = \dfrac{\cos^{n-1}x\sin x}{n} + \dfrac{n-1}{n} I_{n-2}$

6. $$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2} & \text{if } n \text{ is even.} \end{cases}$$

7. $$\int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{1}{2}\dfrac{1}{\pi} & \text{if } n \text{ is even.} \end{cases}$$

8. $$\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{4}{n+5}\cdot\frac{2}{n+3}\left(\frac{1}{n+1}\right), \quad m \text{ is odd, } n \text{ is any case}$$

$$\cancel{\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}} \quad \text{~~m is even, n is even~~}$$

$$\cancel{\int_0^{\pi/2} \sin^m x\cos^n x\,dx =}$$

$m = \text{even}$
$n = \text{even}$

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{3}{n+4} \cdot \frac{1}{n+2} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

$m = \text{even}$
$n = \text{odd}$

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{3}{n+4} \cdot \frac{1}{n+2} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3}$$

---

① Evaluate $\int_0^{\pi/2} \sin^7 x \, dx$

Soln

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \quad \text{when } n \text{ is odd.}$$

$$I_7 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{48}{105}$$

---

② Evaluate $\int_0^{\pi/2} \sin^{10} x \, dx$

Soln

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{when } n \text{ is even.}$$

$$I_{10} = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{63\pi}{512}$$

---

③ Evaluate $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$.

Soln

W.K.T $\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdot \frac{3}{n+4} \cdot \frac{1}{n+2} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

$$\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$=$

③ Evaluate $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$.

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{3}{n+4} \cdot \frac{1}{n+2} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \pi/2$$

$$= \frac{\cancel{5}}{\underset{2}{\cancel{10}}} \cdot \frac{\cancel{3}}{8} \cdot \frac{1}{\underset{2}{\cancel{6}}} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{512}$$

---

④ Evaluate $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx$.

Soln

$$\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \frac{4}{11} \cdot \frac{2}{9} \cdot \frac{1}{7} = \frac{8}{693}$$

---

⑤ Evaluate $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$.

Soln

$$\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx = \frac{4}{12} \cdot \frac{2}{10} \cdot \frac{1}{8} = \frac{1}{120}$$

---

⑥ $\int_0^{\pi/2} \sin^7 x \cos^4 x \, dx$.

Soln

$$\int_0^{\pi/2} \sin^7 x \cos^4 x \, dx = \frac{3}{11} \cdot \frac{1}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$= \frac{16}{1155}$$

Assignment.

Evaluate

① $\int_0^{\pi/2} \sin^9 x\,dx$

② $\int_0^{\pi/2} \cos^8 x\,dx$

③ $\int_0^{\pi/2} \sin^5 x\,dx$

④ $\int_0^{\pi/2} \cos^6 x\,dx$

⑤ $\int_0^{\pi/2} \sin^{10} x\,dx$

⑥ $\int_0^{\pi/2} \cos^9 x\,dx$.

---

② Evaluate

① $\int_0^{\pi/2} \sin^7 x \cos^3 x\,dx$

② $\int_0^{\pi/2} \cos^5 x \sin^4 x\,dx$

③ $\int_0^{\pi/2} \sin^6 x \cos^4 x\,dx$

④ $\int_0^{\pi/2} \cos^6 x \sin^5 x\,dx$

⑤ $\int_0^{\pi/4} \cos^8 x \sin^4 x\,dx$

⑥ $\int_0^{\pi/4} \cos^5 x \sin^5 x\,dx$

Bernoulli's formula

Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Examples

Evaluate $\int e^{ax} \cdot x^3 dx$

Soln

Given $\int e^{ax} \cdot x^3 dx$

Here $u = x^3$ $\qquad dv = e^{ax} dx.$

$u' = 3x^2 \qquad v = \frac{e^{ax}}{a}$

$u'' = 6x \qquad v_1 = \frac{e^{ax}}{a^2}$

$u''' = 6 \qquad v_2 = \frac{e^{ax}}{a^3}$

$v_3 = \frac{e^{ax}}{a^4}.$

$$\int e^{ax} \cdot x^3 dx = uv - u'v_1 + u''v_2 - u'''v_3$$

$$= x^3\left(\frac{e^{ax}}{a}\right) - 3x^2\left(\frac{e^{ax}}{a^2}\right) + 6x\left(\frac{e^{ax}}{a^3}\right) - 6\left(\frac{e^{ax}}{a^4}\right)$$

$$= e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right]$$

$$= \frac{e^{ax}}{a^4}\left[a^3x^3 - 3a^2x^2 + 6ax - 6\right]$$

---

(2) Evaluate $\int x^3 e^x dx$

Soln

Given $\int x^3 e^x dx$.

Here $u = x^3$ $\quad dv = e^x dx$

$u' = 3x^2$ $\quad v = e^x$

$u'' = 6x$ $\quad v_1 = e^x$

$u''' = 6$ $\quad v_2 = e^x$

$v_3 = e^x$.

$\therefore \int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x$$

$$= e^x\left[x^3 - 3x^2 + 6x - 6\right]$$

(3) Evaluate $\int x^2 e^{-2x} dx$.

Soln

Given $\int x^2 e^{-2x} dx$.

$u = x^2$ $\quad dv = e^{-2x} \cdot dx$.

$u' = 2x$ $\quad v = \frac{e^{-2x}}{-2}$

$u'' = 2$ $\quad v_1 = \frac{e^{-2x}}{4}$

$v_2 = \frac{e^{-2x}}{-8}$

$\therefore \int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$$\int x^2 e^{-2x} dx = x^2\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right)$$

$$= e^{-2x}\left[-\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}\right]$$

$$= -\frac{e^{-2x}}{4}\left[2x^2 + 2x + 1\right]$$

④ Evaluate $\int x^4 \sin\frac{x}{2}\,dx$.

Soln

Given $\int x^4 \sin\frac{x}{2}\,dx$

$u = x^4$ $\qquad dv = \sin\frac{x}{2}\,dx$.

$u' = 4x^3$ $\qquad v = \frac{-\cos\frac{x}{2}}{1/2} = -2\cos\frac{x}{2}$

$u'' = 12x^2$ $\qquad v_1 = \frac{-\sin\frac{x}{2}}{1/4} = -4\sin\frac{x}{2}$

$u''' = 24x$ $\qquad v_2 = \frac{\cos\frac{x}{2}}{1/8} = 8\cos\frac{x}{2}$

$u^{iv} = 24$ $\qquad v_3 = \frac{\sin\frac{x}{2}}{1/16} = 16\sin\frac{x}{2}$

$v_4 = \frac{-\cos\frac{x}{2}}{1/32} = -32\cos\frac{x}{2}$.

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + u^{iv}v_4$

$\int x^4 \sin\frac{x}{2}\,dx = x^4\left(-2\cos\frac{x}{2}\right) - 4x^3\left(-4\sin\frac{x}{2}\right) + 12x^2\left(8\cos\frac{x}{2}\right)$

$-24x\left(16\sin\frac{x}{2}\right) + 24\left(-32\cos\frac{x}{2}\right)$

$= -2x^4\cos\frac{x}{2} + 16x^3\sin\frac{x}{2} + 96x^2\cos\frac{x}{2}$

$-384x\sin\frac{x}{2} - 768\cos\frac{x}{2}$

⑤ Evaluate $\int x^3 \sin 3x\,dx$.

Soln

Given $\int x^3 \sin 3x\,dx$

$u = x^3$ $\qquad dv = \sin 3x\,dx$.

$u' = 3x^2$ $\qquad v = -\frac{\cos 3x}{3}$ $\qquad v_2 = \frac{\cos 3x}{27}$

$u'' = 6x$ $\qquad v_1 = -\frac{\sin 3x}{9}$ $\qquad v_3 = \frac{-\sin 3x}{81}$

$u''' = 6$

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$$\int x^3 \sin 3x\,dx = x^3\left(-\frac{\cos 3x}{3}\right) - 3x^2\left(-\frac{\sin 3x}{9}\right) + 6x\left(\frac{\cos 3x}{27}\right) - 6\left(\frac{\sin 3x}{81}\right)$$

$$= -\frac{x^3}{3}\cos 3x + \frac{3x^2}{9}\sin 3x + \frac{6x}{27}\cos 3x - \frac{6}{81}\sin 3x$$

$$= -\frac{x^3}{3}\cos 3x + \frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \frac{2}{27}\sin 3x$$

$$= \frac{1}{27}\left[-9x^3\cos 3x + 9x^2\sin 3x + 6x\cos x - 2\sin 3x\right]$$

$$= \frac{1}{27}\left[(6x - 9x^3)\cos 3x + (9x^2 - 2)\sin 3x\right]$$

---

(6) Evaluate $\int x^5 \cos\frac{x}{2}\,dx$.

Soln

$$\int x^5 \cos\frac{x}{2}\,dx = x^5\left(2\sin\frac{x}{2}\right) - 5x^4\left(-2^2\cos\frac{x}{2}\right) + 20x^3\left(-2^3\sin\frac{x}{2}\right) - 60(x^2)\left(2^4\cos\frac{x}{2}\right) + 120x\left(2^5\sin\frac{x}{2}\right) - 120\left(-2^6\cos\frac{x}{2}\right)$$

$$= 2x\left(x^4 - 80x^2 + 1920\right)\sin\frac{x}{2} + \left(x^4 - 48x^2 + 384\right)20\cos\frac{x}{2}$$

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Homework

① Evaluate $\int x^2 e^x \, dx$.

② Evaluate $\int x e^{3x} \, dx$.

③ Evaluate $\int x^2 e^{-x} \, dx$.

④ Evaluate $\int x^3 e^{-3x} dx$

⑤ Evaluate $\int x^3 \cos 3x \, dx$

⑥ Evaluate $\int x^2 \sin 4x \, dx$

⑦ Evaluate $\int x^3 \cos \frac{x}{3} \, dx$

⑧ Evaluate $\int (x^3+2x^2+3) \sin x \, dx$

⑨ Evaluate $\int e^x (x^3-3x^2+4x-2) dx$.

Evaluate

(7) $\int x^4 \sin x \cos x\, dx$

Soln

$$\int x^4 \sin x \cos x\, dx = \frac{1}{2}\int x^4\, 2\sin x \cos x\, dx$$

$$= \frac{1}{2}\int x^4 \sin 2x\, dx.$$

$$= \frac{1}{2}\left[x^4\left(-\frac{\cos 2x}{2}\right) - (4x^3)\left(-\frac{\sin 2x}{4}\right) + 12x^2\left(\frac{\cos 2x}{8}\right) - 24x\left(\frac{\sin 2x}{16}\right) + 24\left(-\frac{\cos 2x}{32}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{x^4\cos 2x}{2} + x^3 \sin 2x + \frac{3x^2\cos 2x}{2} - \frac{3x\sin 2x}{2} - \frac{3\cos 2x}{4}\right]$$

---

(8) Evaluate $\int x^3 \sin^2 x\, dx.$

Soln

$$\int x^3 \sin^2 x\, dx = \int x^3\left(\frac{1-\cos 2x}{2}\right)dx.$$

$$= \frac{1}{2}\int x^3 dx - \frac{1}{2}\int x^3 \cos 2x\, dx.$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^3\sin 2x}{2} - \frac{3x^2\cos 2x}{4} + \frac{3x\sin 2x}{4} + \frac{3\cos 2x}{8}\right]$$

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Evaluate

(9) $\int_0^{\pi} x^5 \sin x\, dx$

Soln

$\int_0^{\pi} x^5 \sin x\, dx$

$= \Big[x^5(-\cos x) - 5x^4(-\sin x) + 20x^3(\cos x) - 60x^2(\sin x)$
$+ 120x(-\cos x) - 120(-\sin x)\Big]_0^{\pi}$

$= \Big[(-x^5 + 20x^3 - 120x)\cos x + (5x^4 - 60x^2 + 120)\sin x\Big]_0^{\pi}$

$= \Big[(-\pi^5 + 20\pi^3 - 120\pi)\cos\pi\Big]$

$= (-\pi^5 + 20\pi^3 - 120\pi)(-1)$

$= \pi^5 - 20\pi^2 + 120\pi$

---

(10) Evaluate $\int x e^x \sin^2 x\, dx$.

Soln

$\int x e^x \sin^2 x\, dx = \int x \cdot e^x \left(\frac{1-\cos 2x}{2}\right) dx.$

$= \frac{1}{2}\int x e^x dx - \frac{1}{2}\int x e^x \cos 2x\, dx.$

$= \frac{1}{2}\left[x \cdot e^x - 1 \cdot e^x\right] - \frac{1}{2}\Big[x \cdot \frac{e^x}{r}\cos(2x - \theta)$
$- 1 \cdot \frac{e^x}{r^2}\cos(2x - 2\theta)\Big]$

where $r = \sqrt{1^2 + 2^2} = \sqrt{5}$; $\theta = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1}(2)$

$= \frac{e^x}{2}\Big[(x-1) - \frac{x}{\sqrt{5}}\cos(2x - \tan^{-1}2) + \frac{1}{5}\cos(2x - 2\tan^{-1}(2))$

(11) $\int_0^1 x^2 e^{2x} dx$.

soln

$$\int_0^1 x^2 e^{2x} dx = \left[x^2\left(\frac{e^{2x}}{2}\right) - 2x\cdot\left(\frac{e^{2x}}{4}\right) + 2\left(\frac{e^{2x}}{8}\right)\right]_0^1$$

$$= \left[e^{2x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)\right]_0^1$$

$$= e^2\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{4}\right) - e^0\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}e^2 - \frac{1}{4}$$

$$= \frac{1}{4}(e^2 - 1)$$

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(12) Evaluate

$\int x^4 e^x dx$

soln

$$\int x^4 e^x = x^4 e^x - (4x^3) + (12x^2) e^x - (24x) e^x + (24)(e^x)$$

$$= e^x\left[x^4 - 4x^3 + 12x^2 - 24x + 24\right]$$

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(13) Evaluate $\int x^3 e^{-x} dx$

soln

$$\int x^3 e^{-x} dx = x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - 6(e^{-x})$$

$$= e^{-x}\left[(-x^3 - 3x^2 - 6x - 6)\right]$$

$$= -e^{-x}\left[x^3 + 3x^2 + 6x + 6\right]$$

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